Hermite-Padé approximation and location of singularities of systems of analytic functions.

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To my family . . .

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Contents

CHAPTER 1

Introduction

§ 1.1. Padé approximation.

Let $(f_n)_{n\geq 0}$ be a solution of the recurrence relation

$$f_n + \alpha_{n,1} f_{n-1} + \cdots + \alpha_{n,m} f_{n-m} = 0, \qquad n \geq m, \tag{1.1}$$

with given initial values $f_0, \ldots, f_{m-1}$. If for all n the coefficients $\alpha_{n,1}, \ldots, \alpha_{n,m}$ are given, it is well known that the solutions of the recurrence relation form a vector space of dimension $\leq m$.

Recurrence relations play a central role in several areas of mathematics such as number theory, difference equations, continued fractions, and approximation theory to name a few. In the general theory, two results due to H. Poincaré [23] and O. Perron [30, 31] single out (see also [14]).

In the sequel, we assume that the limits

$$\lim_{n\to\infty} \alpha_{n,j} = a_j, \qquad j = 1, \ldots, m, \qquad a_m \neq 0 \tag{1.2}$$

exist. Define the so called characteristic polynomial of (1.1)

$$p(z) = z^m + a_1 z^{m-1} + \cdots + a_m = \prod_{j=1}^{m} (z - \lambda_j). \tag{1.3}$$

***Theorem* 1.1.1 (Poincaré).** *Suppose that $0 < |\lambda_1| < \cdots < |\lambda_m|$. Then, any solution $(f_n)_{n\geq 0}$ of (1.1) verifies that either $f_n = 0, n \geq n_0$, or*

$$\lim_{n\to\infty} \frac{f_{n+1}}{f_n} = \lambda_k, \tag{1.4}$$

where λ_k is one of the roots of the characteristic polynomial.

***Theorem* 1.1.2 (Perron).** *Suppose that* $0 < |\lambda_1| < \cdots < |\lambda_m|$ *and* $\alpha_{n,m} \neq 0, n \geq m$. *Then, there exists a fundamental system of solutions* $(f_n^k)_{n\geq 0}, k = 1, \ldots, m$, *of* (1.1) *such that*

$$\lim_{n\to\infty} \frac{f_{n+1}^k}{f_n^k} = \lambda_k, \qquad k = 1, \ldots, m. \tag{1.5}$$

Each solution $(f_n)_{n\geq 0}$ of (1.1) can be identified with a formal series

$$\mathfrak{f}(z) = \sum_{n\geq 0} f_n z^n. \tag{1.6}$$

We will frequently identify a solution of (1.1) and its associated Taylor series. The analytic properties of an analytic element are encoded in its Taylor coefficients. Due to a deep result of E. Fabry [13], more can be said.

***Theorem* 1.1.3 (Fabry).** *Given a Taylor series* $\mathfrak{f}$ *whose coefficients verify* (1.4) *then* $|\lambda_k|^{-1}$ *is the radius of convergence of* (1.6) *and* λ_k^{-1} *is a singular point of* $\mathfrak{f}$.

Set

$$\alpha_n(z) := 1 + \alpha_{n,1}z + \cdots + \alpha_{n,m}z^m. \tag{1.7}$$

Unless otherwise stated, in the sequel we will assume that $\alpha_{n,m} \neq 0, n \geq m$. If $[\mathfrak{f}]_n$ denotes the n-th taylor coefficient of a formal power series $\mathfrak{f}$, then (1.1) adopts the expression

$$[\mathfrak{f}\alpha_n]_n = 0, \qquad n \geq m. \tag{1.8}$$

It is well known that

$$R_0(\mathfrak{f}) = \left(\limsup_{n\longrightarrow\infty} |f_n|^{1/n}\right)^{-1},$$

$R_0(\mathfrak{f})$ denotes the radius of convergence of the series (1.6) and $D_0(\mathfrak{f}) = \{z : |z| < R_0(\mathfrak{f})\}$ is the disk of convergence. In the sequel, $D_m(\mathfrak{f})$, $m \in \mathbb{Z}_+$, denotes the mth disk of meromorphy of $\mathfrak{f}$. When $R_0(\mathfrak{f}) = 0$ this disk is defined to be the empty set. If $R_0(\mathfrak{f}) > 0$ then $D_m(\mathfrak{f})$ is the largest disk centered at the origin to which the analytic element $(\mathfrak{f}, D_0(\mathfrak{f}))$ can be extended as a meromorphic function having no more than m poles. Let $R_m(\mathfrak{f})$ denote the radius of $D_m(\mathfrak{f})$.

***Definition* 1.1.1.** *Let* $\mathfrak{f}$ *be a formal Taylor expansion about the origin as in* (1.6). *Let* $n, m \in Z_+$, *be non-negative integers. There exists a pair of polynomials* $(p_{n,m}, q_{n,m})$ *such that:*

a.1) $deg(p_{n,m}) \leq n, \quad \deg(q_{n,m}) \leq m, \quad q_{n,m} \not\equiv 0,$

a.2) $q_{n,m}(z)\mathfrak{f}(z) - p_{n,m}(z) = A_{n,m}z^{n+m+1} + \cdots .$

Any pair $(p_{n,m}, q_{n,m})$ *satisfying a.1)-a.2) defines a unique rational function* $\pi_{n,m}(\mathfrak{f}) = p_{n,m}/q_{n,m}$ *called the* (n,m)*-Padé approximant associated with* $\mathfrak{f}$*.*

Thus, to each pair (n, m) you can assign the rational function $\pi_{n,m}(\mathfrak{f})$. In this way, you construct the so called Padé table of rational functions associated with $\mathfrak{f}$. For m fixed, you get the m-th row of the table. Taking $n = m$ for all m you have the main diagonal. Rows and diagonals are the main components of the table. When $m = 0$ you get the Taylor polynomials.

Unless otherwise stated, we will assume that $q_{n,m}$ is normalized with leading coefficient equal to 1.

With this notation, $a.2$) in Definition 1.1.1 reduces to

$$[q_{n,m}\mathfrak{f}]_\nu = 0, \quad \nu = n+1, \cdots, n+m.$$

In terms of recurrence relations, this means that the system of functions

$$(\mathfrak{f}, z\mathfrak{f}, \cdots, z^{m-1}\mathfrak{f}) \tag{1.9}$$

is made up of solutions of a recurrence realtion of type (1.8) (or, what is the same, (1.1)) with α_n replaced by $q_{n,m}$.

One of the main interests in the theory of Padé approximation is that in many cases it allows not only to recover the function but also to detect the location of its singularities. For row sequences, an important result in this direction is due to R. de Montessus de Ballore (see [29]) which we state as follows.

***Theorem* 1.1.4 (Montessus de Ballore).** *Assume that* $\mathfrak{f}$ *has in* $D_m(\mathfrak{f})$ *exactly* m *poles* $\lambda_1, \ldots, \lambda_m$ *(counting multiplicities). Set* $q_m(z) = \prod_{k=1}^{m}(z - \lambda_k)$ *and let* $q_{n,m}$ *denote the denominator of* $\pi_{n,m}$ *normalized with leading coefficient equal to* 1*. Then, for all sufficiently large* n*,* $\deg q_{n,m} = m$*,*

$$\limsup_n \|q_{n,m} - q_m\|^{1/n} \leq \frac{\max\{|\lambda_k| : \ k = 1, \cdots, m\}}{R_m(\mathfrak{f})}, \tag{1.10}$$

where $\|\cdot\|$ *denotes the norm in the space of polynomial coefficients and*

$$\limsup_n \|\pi_{n,m} - \mathfrak{f}\|_{\mathcal{K}}^{1/n} \leq \frac{\max\{|z| : \ z \in \mathcal{K}\}}{R_m(\mathfrak{f})}, \tag{1.11}$$

where $\mathcal{K}$ *denotes an arbitrary compact subset of* $D'_m(\mathfrak{f}) = D_m(\mathfrak{f}) \backslash \{\lambda_1, \cdots, \lambda_m\}$ *and* $\|\cdot\|_{\mathcal{K}}$ *denotes the sup-norm on* $\mathcal{K}$*.*

From Montessus de Ballore's Theorem it follows that if ξ is a pole of $\mathfrak{f}$ in $D_m(\mathfrak{f})$ of order τ, then for each $\epsilon > 0$ sufficiently small, there exists n_0 such that for $n \geq n_0$, $q_{n,m}$ has exactly τ zeros in $\{z : \; |z - \xi| < \epsilon\}$. We say that each pole of $\mathfrak{f}$ in $D_m(\mathfrak{f})$ attracts as many zeros of $q_{n,m}$ as its multiplicity.

We wish to mention that in [6] and [7] similar problems were studied for different types of approximants. For scalar functions, several approximating models have been explored which in one way or another extend the notion of Padé approximation, for example, see [4], [15], [16], [17].

In the theory of Padé approximation the problems may be classified in two groups. In the first group, we have the direct type results in which starting out from a function of which we know some of its analytic properties (for example, region where it is meromorphic, number and location of some of its singularities) we wish to study the convergence of a certain sequence of its Padé approximants (this is the nature of the analytic results you will find in [2]). In the second group, we have the inverse type results where one knows the asymptotic behavior of the poles of a sequence of Padé approximants and one wishes to discover the analytic properties of the analytic element from which the approximants were constructed, and locate some of its singularities. Fabry's theorem is an eloquent example of an inverse type theorem since $(f_n/f_{n+1})_{n\geq 0}$ turns out to be the sequence of poles of the denominators corresponding to the first row of the Padé approximants.

The systematic study of inverse type results was initiated and promoted by A.A. Gonchar in [17].

***Theorem* 1.1.5 (Gonchar).** *Let $\mathfrak{f}$ be a formal Taylor expansion about the origin and fix $m \in \mathbb{N}$. Then, the following two assertions are equivalent:*

a) $R_0(\mathfrak{f}) > 0$ and $\mathfrak{f}$ has exactly m poles in $D_m(\mathfrak{f})$ counting multiplicities.

b) There is a monic polynomial q_m of degree m, $q_m(0) \neq 0$, such that the sequence of denominators $\{q_{n,m}\}_{n\geq m}$ of the Padé approximations of $\mathfrak{f}$, normalized to be monic, satisfies

$$\limsup_{n\to\infty} \|q_m - q_{n,m}\|^{1/n} = \theta < 1,$$

where $\|\cdot\|$ denotes the coefficient norm in the space of polynomials.

Moreover, if either a) or b) takes place the zeros of q_m coincide with the set $\mathcal{P}(\mathfrak{f})$ of poles of $\mathfrak{f}$ in $D_m(\mathfrak{f})$,

$$\theta = \frac{\max\{|\xi| : \xi \in \mathcal{P}_m(\mathfrak{f})\}}{R_m(\mathfrak{f})}, \tag{1.12}$$

and

$$\limsup_{n\to\infty} \|\mathfrak{f} - R_{n,m}\|_K^{1/n} = \frac{\|z\|_K}{R_m(\mathfrak{f})}, \tag{1.13}$$

where K is any compact subset of $D_m(\mathfrak{f}) \setminus \mathcal{P}_m(\mathfrak{f})$.

So stated Gonchar's Theorem first appears as a remark in Section 3, Subsection 4, in [15] (see also [18, Section 2]). Montessus' Theorem is contained in a) implies b) and relations (1.12)-(1.13) with $\leq$ replacing the equality sign. These are the so called direct statements of the theorem. The inverse statements, b) implies a), $\theta \geq \max\{|\xi| : \xi \in \mathcal{P}_m(\mathfrak{f})\}/R_m(\mathfrak{f})$, and the inequality $\geq$ in (1.13) are immediate consequences of [17, Theorem 1]. The study of inverse problems of Padé approximation was suggested by A. A. Gonchar in [17, Subsection 12] where he presented some interesting conjectures. Some of them were solved in [33], [34] and [35] by S. P. Suetin.

A beautiful extension of Fabry's theorem, conjectured in [17], was given by S.P. Suetin in [34] under the assumption that the sequence of denominators of the m-th row of the Padé approximants is convergent.

***Theorem* 1.1.6 (Suetin).** *Suppose that $m \in \mathbb{N}$ is fixed and (1.6) is a power series which is not a rational function having at most $m-1$ poles. Assume that there exists a polynomial $q_m(z) = \prod_{k=1}^m (z - \lambda_k)$, $q_m(0) \neq 0$, such that*

$$\lim_{n\longrightarrow\infty} q_{n,m}(z) = q_m(z)$$

where $\{q_{n,m}\}$, $n \geq 1$, is the sequence of denominators of the m-th row of Padé approximants of (1.6). Then

- *a) $R_{m-1}(\mathfrak{f}) = \max_{1\leq k\leq m} |\lambda_k|$. All the points λ_k $(k = 1, \cdots, m)$ lying in the disk $|z| < R_{m-1}(\mathfrak{f})$, and only they, are poles of $\mathfrak{f}$ (counting multiplicity) in this disk.*
- *b) All the points λ_k $(k = 1, \cdots, m)$ lying on the circle $|z| = R_{m-1}(\mathfrak{f})$ are singular points of $\mathfrak{f}$.*

§ 1.2. Hermite-Padé approximation.

In [19] Ch. Hermite publishes his proof of the transcendence of e making use of simultaneous rational approximation of systems of exponentials. That paper marked the beginning of the modern analytic theory of numbers. The formal theory of simultaneous rational approximation for general systems of analytic functions, now called Hermite-Padé approximation, was initiated by K. Mahler in lectures delivered at the University of Groningen in 1934-35.

These lectures were published years later in [27]. Important contributions in this respect are due to his students J. Coates and H. Jager, see [11] and [20]. Basically, there are two types of Hermite-Padé approximants, called of type I and II. We restrict our attention to type II Hermite-Padé approximants because they are the ones related with higher order recurrence relations.

Let $\mathbf{f} = (\mathfrak{f}^1, \mathfrak{f}^2, \ldots, \mathfrak{f}^d)$ be a system of d formal series, where for each $k = 1, \ldots, d$,

$$\mathfrak{f}^k(z) = \sum_{n=0}^{\infty} f_n^k z^n, \qquad f_n^k \in \mathbb{C}. \tag{1.14}$$

Let $\mathbf{D} = (D_1, D_2, \ldots, D_d)$ be a system of domains such that, for each $k = 1, \ldots, d$, $\mathfrak{f}^k$ is meromorphic in D_k. We say that the point ξ is a pole of $\mathbf{f}$ in $\mathbf{D}$ of order τ if there exists an index $k \in 1, \ldots, d$ such that $\xi \in D_k$ and it is a pole of $\mathfrak{f}^k$ of order τ, and for $j \neq k$ either ξ is a pole of $\mathfrak{f}^j$ of order less than or equal to τ or $\xi \notin D_j$. When $\mathbf{D} = (D, \ldots, D)$ we say that ξ is a pole of $\mathbf{f}$ in D.

Let $R_0(\mathbf{f})$ be the radius of the largest open disk $D_0(\mathbf{f})$ in which all the expansions $\mathfrak{f}^k$, $k = 1, \ldots, d$ correspond to analytic functions. If $R_0(\mathbf{f}) = 0$, we take $D_m(\mathbf{f}) = \emptyset$, $m \in \mathbb{Z}_+$; otherwise, $R_m(\mathbf{f})$ is the radius of the largest open disk $D_m(\mathbf{f})$ centered at the origin to which all the analytic elements $(\mathfrak{f}^k, D_0(\mathfrak{f}^k))$ can be extended so that $\mathbf{f}$ has at most m poles counting multiplicities. The disk $D_m(\mathbf{f})$ constitutes for systems of functions the analogue of the m-th disk of meromorphy defined by J. Hadamard in [25] for $d = 1$. Moreover, in that case both definitions coincide.

In the sequel, $\mathbf{m} = (m_1, \ldots, m_d) \in \mathbb{Z}_{\geq 0}^d \setminus \{\mathbf{0}\}$ is fixed. Here, $\mathbb{Z}_{\geq 0} := \{0, 1, 2, \ldots\}$ and $\mathbf{0} \in \mathbb{Z}_{\geq 0}^d$ denotes the zero vector. Set $|\mathbf{m}| = m_1 + \cdots + m_d$.

***Definition* 1.2.1.** *Let* $(\mathbf{f}, \mathbf{m})$ *and* $n \geq \max\{m_k : k = 1, \ldots, d\}$ *be given. Then, there exist polynomials* q, p_k, $k = 1, \ldots, d$, *such that*

b.1) $\deg p_k \leq n - m_k$, $k = 1, \ldots, d$, $\deg q \leq |\mathbf{m}|$, $q \not\equiv 0$,

b.2) $(q\mathfrak{f}^k - p_k)(z) = A_k z^{n+1} + \cdots$.

The vector rational function $\mathbf{R}_{n,\mathbf{m}} = (R_{n,\mathbf{m},1}, \ldots, R_{n,\mathbf{m},d})$ *where* $R_{n,\mathbf{m},k} = p_k/q$ *is called an* $(n, \mathbf{m})$ *(type II) Hermite-Padé approximation of* $\mathbf{f}$.

The existence of such polynomials reduces to solving a homogeneous linear system of $|\mathbf{m}|$ equations on the $|\mathbf{m}|+1$ unknown coefficients of q. Thus a nontrivial solution exists. Once q is found, the polynomial $p_k, k = 1, \ldots, d$, is the Taylor polynomial of degree $n - m_k$ of $q\mathfrak{f}^k$. Hermite-Padé approximants are not uniquely determined in general, except when $d = 1$. For each n we choose one candidate.

Without loss of generality, we can assume that q has no common zero simultaneously with all the polynomials p_k except possibly at $z = 0$. Indeed, if $z_0 \neq 0$ was such a common zero, we can divide both sides of $b.2$) by $z - z_0$ lowering the degrees of q and p_k while preserving the starting power on the right hand of $b.2$). This last observation cannot be achieved if the common zero is $z_0 = 0$. We can take the leading coefficient of q equal to 1. With these normalization, we write $q_{n,\mathbf{m}}$ and $p_{n,\mathbf{m},k}$ instead of q and p_k, respectively.

With this notation, $b.2$) in Definition 1.2.1 reduces to

$$[z^\nu q_{n,\mathbf{m}} \mathfrak{f}^k]_n = 0, \qquad k = 1, \ldots, d, \qquad \nu = 0, \ldots, m_k - 1.$$

In terms of recurrence relations, this means that the system of functions

$$(\mathfrak{f}^1, \ldots, z^{m_1-1}\mathfrak{f}^1, \mathfrak{f}^2, \ldots, z^{m_2-1}\mathfrak{f}^2, \mathfrak{f}^3, \ldots, z^{m_d-1}\mathfrak{f}^d) \tag{1.15}$$

is made up of solutions of the recurrence relations

$$[q_{n,\mathbf{m}}\mathfrak{f}]_n = 0, \qquad n \geq |\mathbf{m}|, \tag{1.16}$$

of type (1.8) (or, what is the same, (1.1)) with α_n replaced by $q_{n,\mathbf{m}}$. We cannot guarantee that the independent term or the coefficient accompanying $z^{|\mathbf{m}|}$ of $q_{n,\mathbf{m}}$ are different from zero, but this will not cause any problem for the applications we have in mind.

On the other hand, given the recurrence (1.1), let $\mathbf{f} = (\mathfrak{f}^1, \ldots, \mathfrak{f}^m)$ be a fundamental system of solutions of (1.1) and take $\mathbf{m} = (1, 1, \ldots, 1)$, where 1 is repeated m times. It is easy to verify that α_n is non other than the common denominator $q_{n,\mathbf{m}}$ of the $(n, \mathbf{m})$ type II Hermite Padé approximation of $\mathbf{f}$.

Returning to the setting of Hermite-Padé approximation, when $d = 1$, then $\mathbf{f} = \mathfrak{f}$ reduces to a scalar function, $\mathbf{m} = m$ is a non-negative integer, and Definition 1.2.1 gives rise to the so called $(n - m, m)$ Padé approximation of $\mathfrak{f}$.

By construction, all the polynomials $q_{n,\mathbf{m}}$ have degree $\leq |\mathbf{m}|$. The sequence $(\mathbf{R}_{n,\mathbf{m}})_{n \geq n_0}$ of vector rational functions, where $\mathbf{m}$ remains fixed, is called a row sequence of Hermite-Padé approximants of $\mathbf{f}$ in consonance with the denomination established in the scalar case $d = 1$ in the theory of Padé approximation. One can consider sequences of type II Hermite-Padé approximations in which $\mathbf{m}$ depends on n, but this drives us far away from the connection established above with the theory of recurrence relations, specially if $|\mathbf{m}|$ increases to ∞ with n.

It should be stressed that [24] was pioneering in the sense that it initiated a convergence theory for row sequences of Hermite- Padé approximation. The result proved in [24] does not allow a converse statement in the sense of Gonchar's Theorem. Inspired in the concept of polewise independence, in [10] the following relaxed version of it was introduced.

***Definition* 1.2.2.** *Given* $\mathbf{f} = (\mathfrak{f}^1, \ldots, \mathfrak{f}^d)$ *and* $\mathbf{m} = (m_1, \ldots, m_d) \in \mathbb{Z}^d_{\geq 0} \setminus \{\mathbf{0}\}$ *we say that* $\zeta \in \mathbb{C} \setminus \{0\}$ *is a system pole of order* τ *of* $(\mathbf{f}, \mathbf{m})$ *if* τ *is the largest positive integer such that for each* $s = 1, \ldots, \tau$ *there exists at least one polynomial combination of the form*

$$\sum_{k=1}^{d} p_k \mathfrak{f}^k, \qquad \deg p_k < m_k, \qquad k = 1, \ldots, d, \tag{1.17}$$

which is analytic on a neighborhood of $\overline{D}_{|\zeta|} = \{z : |z| \leq |\zeta|\}$ *except for a pole at* $z = \zeta$ *of exact order* s*. If some component* m_k *equals zero the corresponding polynomial* p_k *is taken identically equal to zero.*

We wish to underline that if some component m_k equals zero, that component places no restriction on Definition 1.2.1 and does not report any benefit in finding system poles; therefore, without loss of generality one can restrict the attention to multi-index $\mathbf{m} \in \mathbb{N}^d$.

***Definition* 1.2.3.** *Given* $\mathbf{f} = (\mathfrak{f}^1, \ldots, \mathfrak{f}^d)$ *and* $\mathbf{m} = (m_1, \ldots, m_d) \in \mathbb{Z}^d_{\geq 0} \setminus \{\mathbf{0}\}$ *we say that* $\zeta \in \mathbb{C} \setminus \{0\}$ *is a system singularity of* $(\mathbf{f}, \mathbf{m})$ *if there exists at least one polynomial combination of the form* (1.17) *analytic in* $D_{|\zeta|} = \{z : |z| < |\zeta|\}$ *and* ζ *is a singular point of* (1.17).

In this context, the concepts of singular point and pole depend not only on the system of functions $\mathbf{f}$ but also on the multi index $\mathbf{m}$. For example, poles of the individual functions $\mathfrak{f}^k$ need not be system poles of $\mathbf{f}$ and system poles need not be poles of any of the functions $\mathfrak{f}^k$ (see interesting examples in [10]).

The example of Table 5.3 shows that given $(\mathbf{f}, \mathbf{m})$ a point in $\mathbb{C}^*$ may be simultaneously a system pole and a singularity of a different nature. Obviously, 1 is a system pole of $(\mathbf{f}, \mathbf{m})$ of order one because of $\mathfrak{f}_2$, and it is also a system syngularity of logarithmic type because of $\mathfrak{f}_1$.

***Definition* 1.2.4.** *A vector* $\mathbf{f} = (\mathfrak{f}^1, \ldots, \mathfrak{f}^d)$ *of* d *formal Taylor expansions is said to be polynomially independent with respect to* $\mathbf{m} = (m_1, \ldots, m_d) \in \mathbb{N}^d, \mathbb{N} := \{1, 2, \ldots\}$, *if there do not exist polynomials* $p_1, \ldots, p_d$, *at least one of which is non-zero, such that*

b.1) $\deg p_k < m_k$, $k = 1, \ldots, d$,

b.2) $\sum_{k=1}^{d} p_k \mathfrak{f}^k$ *is a polynomial.*

In particular, polynomial independence implies that for $k = 1, \dots, d$, $\mathfrak{f}^k$ is not a rational function with at most $m_k - 1$ poles and the system of functions (1.15) is linearly independent. Moreover, (1.15) constitutes a fundamental system of solutions of (1.16).

To each system pole ξ of $\mathbf{f}$ with respect to $\mathbf{m}$ one can associate several characteristic values. Let τ be the order of ξ as a system pole of $\mathbf{f}$. For each $s = 1, \dots, \tau$ denote by $r_{\xi,s}(\mathbf{f}, \mathbf{m})$ the largest of all the numbers $R_s(g)$ (the radius of the largest disk containing at most s poles of g), where g is a polynomial combination of type (1.17) that is analytic on a neighborhood of $\overline{D}_{|\xi|}$ except for a pole at $z = \xi$ of order s. Set

$$R_{\xi,s}(\mathbf{f}, \mathbf{m}) := \min_{k=1,\dots,s} r_{\xi,k}(\mathbf{f}, \mathbf{m}),$$

$$R_{\xi}(\mathbf{f}, \mathbf{m}) := R_{\xi,\tau}(\mathbf{f}, \mathbf{m}) := \min_{s=1,\dots,\tau} r_{\xi,k}(\mathbf{f}, \mathbf{m}).$$

It is not difficult to verify that if $d = 1$ and $(\mathbf{f}, \mathbf{m}) = (\mathfrak{f}, m)$, the concepts of system poles and poles in $D_m(\mathfrak{f})$ coincide.

Let $\mathcal{Q}(\mathbf{f}, \mathbf{m})$ denote the monic polynomial whose zeros are the system poles of $\mathbf{f}$ with respect to $\mathbf{m}$ taking account of their order. The set of distinct zeros of $\mathcal{Q}(\mathbf{f}, \mathbf{m})$ is denoted by $\mathcal{P}(\mathbf{f}, \mathbf{m})$.

Gonchar's theorem was extended to the context of row sequences of type II Hermite-Padé approximation in [10, Theorem 1.4]. Taking into consideration the connection with the general theory of recurrence relations established above, this result may be reformulated in terms of the analytic continuation and singularities of a fundamental system of solutions of a general recurrence relation.

***Theorem* 1.2.1.** *Let* $\mathbf{f}$ *be a system of formal Taylor expansions as in* (1.14) *and fix a multi-index* $\mathbf{m} \in \mathbb{N}^d$*. Then, the following assertions are equivalent.*

a) $R_0(\mathbf{f}) > 0$ *and* $\mathbf{f}$ *has exactly* $|\mathbf{m}|$ *system poles with respect to* $\mathbf{m}$ *counting multiplicities.*

b) The denominators $q_{n,\mathbf{m}}$*,* $n \geq |\mathbf{m}|$*, of simultaneous Padé approximations of* $\mathbf{f}$ *are uniquely determined for all sufficiently large* n *and there exists a polynomial* $q_{\mathbf{m}}$ *of degree* $|\mathbf{m}|$*,* $q_{\mathbf{m}}(0) \neq 0$*, such that*

$$\limsup_{n \longrightarrow \infty} \|q_{\mathbf{m}} - q_{n,\mathbf{m}}\|^{1/n} = \theta < 1.$$

Moreover, if either a) or b) takes place then $q_{\mathbf{m}} \equiv \mathcal{Q}(\mathbf{f}, \mathbf{m})$ *and*

$$\theta = \max\{\frac{|\xi|}{R_{\xi}(\mathbf{f}, \mathbf{m})} : \ \xi \in \mathcal{P}(\mathbf{f}, \mathbf{m})\}. \tag{1.18}$$

If $d = 1$, $R_{n,m}$ and $q_{n,m}$ are uniquely determined; therefore, Theorem 1.2.1 contains Gonchar's Theorem.

Inspired in the conjectures posed by A.A. Gonchar in [17] for the scalar case, some natural questions arise. Is it true that each system pole attracts with geometric rate at least as many zeros of the polynomials $q_{n,\mathbf{m}}$ as its order (even if the total number of system poles is less than $|\mathbf{m}|$)? Reciprocally, if some point in $\mathbb{C}^*$ attracts a certain number of zeros of the polynomials $q_{n,\mathbf{m}}$ with geometric rate, does it mean that the point is necessarily a system pole of $(\mathbf{f}, \mathbf{m})$? What can be said about the points which are limit of the zeros of the denominators? Are they singular points of $(\mathbf{f}, \mathbf{m})$ in some sense? In this regards, the numerical results in Chapter 5 make us suspect that the answer is yes.

We wish to investigate the case when

$$\lim_{n \longrightarrow \infty} q_{n,\mathbf{m}} = q_{\mathbf{m}}, \qquad \deg q_{\mathbf{m}} = |\mathbf{m}|, \qquad q_{\mathbf{m}}(0) \neq 0, \tag{1.19}$$

but the rate of convergence is not known in advance.

Assuming (1.19), one of the goal of this thesis is to study the connection between the zeros of $q_{\mathbf{m}}$ and the system singularities of $(\mathbf{f}, \mathbf{m})$ which would give a generalization of Suetin's Theorem.

In the proof of Theorem 1.2.1 the concept of incomplete Padé approximation plays a central role. This notion was introduced in [9]. We will further explore its potential for our purpose. In the next section we give its formal definition and mention some of its properties.

§ 1.3. Incomplete Padé approximation.

We begin with a formal definition of this concept.

***Definition* 1.3.1.** *Let $\mathfrak{f}$ be a formal Taylor expansion about the origin. Fix $m \geq m^* \geq 1$. Let $n \geq m$, we say that the rational function $r_{n,m}$ is an incomplete Padé approximation of type (n, m, m^*) with respect to $\mathfrak{f}$ if $r_{n,m}$ is the quotient of any two polynomials p and q that verify*

(c.1) $deg(p) \leq n - m^*$, $\quad deg(q) \leq m \quad q \not\equiv 0$,

(c.2) $q(z)\mathfrak{f}(z) - p(z) = Az^{n+1} + \cdots$.

Given (n, m, m^*), $n \geq m \geq m^*$, the Padé approximants $R_{n-m^*,m^*}, \ldots, R_{n-m^*,m}$ can all be regarded as incomplete Padé approximation of type (n, m, m^*) of $\mathfrak{f}$. In particular this means that $r_{n,m}$ is not uniquely determined (in general) when $m^* < m$. Therefore, when we refer to such approximants we understand that once we fix m and m^* for each given n a candidate is chosen.

This liberty is the main convenience of incomplete Padé approximation. For example, notice that according to the definition of Hermite Padé approximation $R_{n,\mathbf{m},k}$ is an incomplete Padé approximation of type $(n, |\mathbf{m}|, m_k)$ of the kth component $\mathfrak{f}^k$ of the vector $\mathbf{f}$. In Chapter 5, the components $R_{n,\mathbf{m},k}$ are incomplete Padé approximations of type $(n, 2, 1)$ in the examples of Tables 5.1, 5.2 and 5.3, while in the other examples they are incomplete Padé approximations of type $(n, 3, 1)$.

Cancelling out common factors between p and q, we write $r_{n,m} = p_{n,m}/q_{n,m}$, where $q_{n,m}$ is normalized as follows

$$q_{n,m}(z) = \prod_{|\zeta_{n,k}|<1} (z - \zeta_{n,k}) \prod_{|\zeta_{n,k}|\geq 1} (1 - z/\zeta_{n,k}) \tag{1.20}$$

With this normalization, it is easy to check that on any compact subset $\mathcal{K}$ of $\mathbb{C}$

$$\|q_{n,m}\|_{\mathcal{K}} := \max_{z\in\mathcal{K}} |q_{n,m}(z)| \leq C < \infty \tag{1.21}$$

where C is a constant that is independent of $n \in \mathbb{N}$ (but depends on $\mathcal{K}$).

Suppose that p and q have a common zero at $z = 0$ of order λ_n. Notice that $0 \leq \lambda_n \leq m$. Then

(c.3) $\deg p_{n,m} \leq n - m^* - \lambda_n, \quad \deg q_{n,m} \leq m - \lambda_n, \quad q_{n,m} \not\equiv 0$

(c.4) $q_{n,m}(z)\mathfrak{f}(z) - p_{n,m}(z) = Az^{n+1-\lambda_n} + \cdots$.

From the definition it is not hard to prove (see proposition 2.1.1) that

$$r_{n+1,m} - r_{n,m} = \frac{A_{n,m} z^{n+1-\lambda_n-\lambda_{n+1}} q^*_{n,m-m^*}}{q_{n,m} q_{n+1,m}}, \tag{1.22}$$

where $A_{n,m}$ is a constant and $q^*_{n,m-m^*}$ is a polynomial of degree less than or equal to $m - m^*$ normalized as in (1.20).

We introduce a notion of convergence which will be very useful in the sequel.

***Definition* 1.3.2.** *Let E be a subset of the complex plane $\mathbb{C}$. By $\mathcal{U}(E)$ we denote the class of all coverings of E by at most a numerable set of disks. Set*

$$\sigma_1(E) := inf \left\{ \sum_{\nu=1}^{\infty} |U_\nu| : \ \{U_\nu\} \in \mathcal{U}(E) \right\}$$

where $|U_\nu|$ denotes the radius of the disk U_ν.

The quantity $\sigma_1(E)$ is called the σ_1 content of the set E. The following properties are immediate consequences of the definition.

b.1) $\sigma_1(E_1 \cup E_2) \leq \sigma_1(E_1) + \sigma_1(E_2)$.

b.2) if $E_1 \subset E_2$ then $\sigma_1(E_1) \leq \sigma_1(E_2)$.

Obviously, if E is itself a (closed or open) disk then $\sigma_1(E)$ is equal to its radius. We have the following concept of convergence.

***Definition* 1.3.3.** *Let φ and φ_n, $n \in \mathbb{Z}_+$, be functions defined on a region $\Omega \subset \mathbb{C}$. We say that the sequence $(\varphi_n)_{n\geq 0}$ converges σ_1 on each compact subset $K \subset \Omega$ to φ if for every $K \subset \Omega$ and $\epsilon > 0$*

$$\lim_{n\longrightarrow\infty} \sigma_1 \{z \in K : \ |(\varphi_n - \varphi)(z)| \geq \epsilon\} = 0.$$

We denote this by

$$\sigma_1 - \lim_n \varphi_n = \varphi, \quad K \subset \Omega.$$

Using telescopic sums, equation (1.22) implies that σ_1 convergence of the sequence $(r_{n,m})_{n\geq 0}$ can be reduced to the σ_1 convergence of the series

$$\sum_{n=m}^{\infty} \frac{A_{n,m} z^{n+1-\lambda_n-\lambda_{n+1}} q^*_{n,m-m^*}(z)}{(q_{n,m}q_{n+1,m})(z)}, \quad 0 \leq \lambda_n \leq m.$$

Define

$$R^*_m(f) = \frac{1}{\limsup\limits_{n\longrightarrow\infty} |A_{n,m}|^{1/n}}, \qquad D^*_m(f) = \{z : \ |z| < R^*_m(f)\}. \tag{1.23}$$

Among other results, in [9] the authors proved

***Theorem* 1.3.1.** *Let $\mathfrak{f}$ be a formal power series. Fix m and m^* nonnegative integers, $m \geq m^*$. Let $(r_{n,m})_{n\geq m}$ be a sequence of incomplete Padé approximants of type (n, m, m^*) for f. If $R^*_m(\mathfrak{f}) > 0$ then $R_0(\mathfrak{f}) > 0$. Moreover,*

$$D_{m^*}(\mathfrak{f}) \subset D^*_m(\mathfrak{f}) \subset D_m(\mathfrak{f})$$

and $D^*_m(\mathfrak{f})$ is the largest disk in compact subsets of which $\sigma_1 - \lim\limits_{n\to\infty} r_{n,m} = \mathfrak{f}$. Moreover, the sequence $(r_{n,m})_{n\geq m}$ is pointwise divergent in $\{z : \ |z| > R^*_m(\mathfrak{f})\}$ except on a set of σ_1-content zero.

When dealing with inverse type problems, one of the main difficulties is to determine from the data if the formal expansion represents an analytic element in a vicinity of the origin; that is, if the formal expansion is indeed convergent about $z = 0$. The previous theorem says that a sufficient condition is that $R^*_m(\mathfrak{f}) > 0$. Notice that in that result the convergence of the denominators of the incomplete Padé approximants is not required. When this is true some additional information can be drawn. A direct consequence of [10, Corollary 2.4] establishes

***Theorem* 1.3.2.** *Let $\mathfrak{f}$ be a formal power series that is not a polynomial. Fix $m \geq m^* \geq 1$. Let $(r_{n,m})_{n \geq m}, r_{n,m} = p_{n,m}/q_{n,m}$, be a sequence of incomplete Padé approximants of type (n, m, m^*) corresponding to $\mathfrak{f}$. Assume that there exists a polynomial q_m of degree m, $q_m(0) \neq 0$, such that*

$$\lim_{n \to \infty} q_{n,m} = q_m. \tag{1.24}$$

Then, $0 < R_0(\mathfrak{f}) < \infty$ and the zeros of q_m contain all the poles, counting multiplicities, that $\mathfrak{f}$ has in $D_m^*(\mathfrak{f})$.

Therefore, incomplete Padé approximation allows to recover the poles of $\mathfrak{f}$ inside $D_m^*(\mathfrak{f})$. When $m^* = m$ we are in the case of Padé approximation and Suetin's Theorem says that all the zeros of q_m are singular points of $\mathfrak{f}$, whether they lie in $D_m(\mathfrak{f})$ or its boundary. For truly incomplete Padé approximants ($m^* < m$), what can be said about the zeros of q_m in relation with the singular points of $\mathfrak{f}$? We know that not all of them need to be singular points as can be deduced from the examples in [9, Section 5]. However, $\mathfrak{f}$ may have less than m^* poles in $D_m^*(\mathfrak{f})$, we can see that clearly in the example of Table 5.3, where $R_1(\mathfrak{f}_1) = R_2^*(\mathfrak{f}_1) = R_2(\mathfrak{f}_1) = 1$ and $\mathfrak{f}_1$ has no poles in $D_2^*(\mathfrak{f}_1)$. In this situation, do the zeros of q_m contain some singularities of $\mathfrak{f}$ lying on the boundary of $D_m^*(\mathfrak{f})$? This and other related questions will be addressed in this thesis.

In Chapter 5 we show some examples which make us wonder the following:

- Is it true or false that
$$\lim_{n \longrightarrow \infty} q_{n,m} = q_m \tag{1.25}$$
implies that
$$\lim_{n \longrightarrow \infty} q^*_{n,m-m^*} = q^*_{m-m^*} \tag{1.26}$$

- Suppose that (1.25) and (1.26) hold. Is it true or false that q_m is divisible by $q^*_{m-m^*}$?

§ 1.4. Multipoint Padé approximation.

Suppose that E is a bounded continuum with connected complement containing more than one point, $\alpha = \{\alpha_{n,k}\}$ $(n = 1, 2, \ldots; \ k = 1, \ldots, n)$ is a table of interpolation nodes.

***Definition* 1.4.1.** *Let $\mathfrak{f}(z) \in \mathcal{H}(E)$. Then, there exist polynomials $P_{n,m}$, $Q_{n,m}$, such that*

d.1) $\deg P_{n,m} \leq n$, $\deg Q_{n,m} \leq m$, $Q_{n,m} \not\equiv 0$, *d.2)* $(Q_{n,m}\mathfrak{f} - P_{n,m})/a_{n+m+1} \in \mathcal{H}(E)$,

where $a_n(z) = \prod_{k=1}^{n}(z-\alpha_{n,k})$*. A multipoint Padé approximant of type* (n,m) *for the function* $\mathfrak{f}(z)$ *is defined to be a rational function given as the ratio* $P_{n,m}/Q_{n,m}$ *of any plynomials* $P_{n,m}$ *and* $Q_{n,m}$ *satisfying* $d.1) - d.2)$*.*

In the study of the convergence of general interpolation schemes, it is common to impose on the table of interpolation nodes various restrictions which determine the asymptotic behavior of the sequence of polynomials a_n. Let Φ_E be a holomorphic univalent function mapping the complement of E onto the exterior of the unit disk, with $\Phi_E(\infty) = \infty, \Phi'_E(\infty) > 0$. It is well known that there exist tables of points α satisfying the condition

$$\lim_{n\to\infty} |a_n(z)|^{1/n} = c|\Phi_E(z)|, \tag{1.27}$$

or the stronger condition

$$\lim_{n\to\infty} a_n(z)/c^n\Phi_E^n(z) = G(z) \neq 0, \tag{1.28}$$

uniformly on compact subsets of $\overline{\mathbb{C}} \setminus E$, where c denotes some positive constant, see [37, Chapters 8-9]. For each $\rho > 1$, we introduce

$$\Gamma_\rho := \{z \in \mathbb{C} : |\Phi_E(z)| = \rho\}, \quad \text{and} \quad D_\rho := E \cup \{z \in \mathbb{C} : |\Phi_E(z)| < \rho\}$$

as the *level curve of index* ρ and the *canonical domain of index* ρ, respectively. Let $\rho_0(\mathfrak{f})$ be equal to the index ρ of the largest canonical domain D_ρ to which $\mathfrak{f}$ can be extended as holomorphic functions, and $\rho_m(\mathfrak{f})$ is the largest number ρ such that $\mathfrak{f}$ admits meromorphic extension to the domain D_ρ and has at most m poles in this domain.

Gonchar proved that the following analogue of the Cauchy-Hadamard formula holds for $\mathfrak{f} \in \mathcal{H}(E)$ and interpolation tables satisfying (1.28):

$$\rho_0(\mathfrak{f})^{-1} = c \limsup_{n\to\infty} \left| \int_{\Gamma_\rho} a_{n+1}(t)^{-1}\mathfrak{f}(t)dt \right|^{1/n}. \tag{1.29}$$

With the help of this formula he obtained the following theorem.

***Theorem* 1.4.1 (Gonchar).** *Suppose that* $\mathfrak{f}(z) \in \mathcal{H}(E)$*, the interpolation nodes satisfy condition* (1.28)*, and for all sufficiently large* n *the* m*-th row of the table of multipoint Padé approximants has exactly* m *finite poles* $\lambda_{n,1}, \ldots, \lambda_{n,m}$*, which converge to limits* $\lambda_1, \ldots, \lambda_m$ *at the rate of a geometric progression:* $\limsup_{n\to\infty} |\lambda_{n,p} - \lambda_p|^{1/n} = \delta_p < 1$ *(*$p = 1, \cdots, m$*). Then* $\rho_m(\mathfrak{f}) = \delta^{-1}|\Phi_E(\lambda_p)|$ *(*$p = 1, \cdots, m$*), and all the points* $\lambda_1, \ldots, \lambda_m$ *(and only they) are poles of* $\mathfrak{f}(z)$ *(counting multiplicity) in* $D_{\rho_m(\mathfrak{f})}$*.*

In [3] the author shows that the conditions characterizing the rate of convergence of the poles can be waived in Gonchar's Theorem. More precisely, he proves the following analogue of Suetin's Theorem for the m-th row of the table of multipoint Padé approximants of a function $\mathfrak{f}(z)$ holomorphic in a neighborhood of a continuum E.

***Theorem* 1.4.2.** *Suppose that $\mathfrak{f}(z) \in \mathcal{H}(E)$, the interpolation nodes satisfy condition* (1.28)*, and for all sufficiently large n the m-th row of the table of multipoint Padé approximants has exactly m finite poles $\lambda_{n,1}, \ldots, \lambda_{n,m}$, which converge to limits $\lim_{n\to\infty} \lambda_{n,p} = \lambda_p$ $(p = 1, \cdots, m)$. Then $\rho_{m-1}(\mathfrak{f}) = \max_{1\leq p\leq m} |\Phi_E(\lambda_p)|$, all the points λ_p $(p = 1, \cdots, m)$ lying on the boundary of $D_{\rho_{m-1}(\mathfrak{f})}$ are singular points of $\mathfrak{f}(z)$.*

In Chapter 4 we gived necessary and sufficient conditions for the convergence with geometric rate of the common denominators of multipoint Hermite-Padé approximants.

§ 1.5. Methodology and structure of the thesis.

In terms of the asymptotic behavior of the sequence of common denominators, in Chapter 2 we describe some analytic properties of $\mathbf{f}$ and restate some conjectures corresponding to questions once posed by A. A. Gonchar for row sequences of Padé approximants. The main result of this chapter is Theorem 2.3.1 contained in Section 2.3. Its proof is based on two fundamental lemmas proved in Section 2.2. Theorem 2.3.1 is an extension of Suetin's Theorem for the case of incomplete Padé approximation. Actually, when $m^* = m$, Theorem 2.3.1 reduces to Theorem 1.1.6. In Section 2.4 we prove that the system poles are strong attractors. In Section 2.5 we consider row sequences of (type II) Hermite-Padé approximations with common denominator associated with a vector $\mathbf{f}$ of formal power expansions about the origin and we apply Theorem 2.3.1 to Hermite-Padé approximation.

In Chapter 3 we obtain extensions of the Poincaré and Perron theorems for higher order recurrence relations. The main result is Theorem 3.2.1 contained in Section 3.2. In Section 3.3 we prove some auxiliary lemmas needed for the proof of Theorem 3.2.1, which is in Section 3.4. We obtain some consequences of Theorem 3.2.1 in the study of row sequences of Hermite-Padé approximation contained in Section 3.5.

In Chapter 4, given a system of functions $\mathbf{f} = (\mathfrak{f}_1, \ldots, \mathfrak{f}_d)$ analytic on a neighborhood of some compact subset E of the complex plane, we give necessary and sufficient conditions for the convergence with geometric rate of the common denominators of (MHP) multipoint Hermite-Padé approximants. The main result of this chapter is the Theorem 4.1.1, which extends

Theorem 1.2.1 to the case of MHP approximation. We prove the direct and inverse statements in Sections 4.2 and 4.3 respectively. The exact rate of convergence of the denominators and of the approximants themselves is given in terms of the analytic properties of the system of functions. These results allow to detect the location of the poles of the system of functions which are in some sense "closest" to E.

Chapter 5 contains some computational experiments to illustrate the behavior of the denominators of Hermite-Padé approximants and the relationship between the zeros of q_m and $q^*_{m-m^*}$. For Padé approximants this problem does not arise because $q^*_{m-m^*} \equiv 1$ since $m = m^*$.

Some of the results of this thesis were announced without proofs in [26]. That paper served as guidance and inspiration for the research we have carried out. The contents of Chapter 2 appears in [21]. The contents of Chapter 3 is contained in [22]. The contents of Chapter 4 was submitted and accepted for publication in [8].

CHAPTER 2

Direct and inverse results on row sequences of Hermite-Padé approximation

§ 2.1. Some auxiliary results.

We begin proving formula (1.22) which plays a central role in our reasonings.

***Proposition* 2.1.1.** *Let a formal power series* (1.6) *be given. Fix* $m \geq m^*$ *two positive integers. Then, for each* $n \geq m$ (1.22) *takes place.*

Proof. Using (c.4) on page 12 we have

$$z^{\lambda_n}[q_{n,m}\mathfrak{f} - p_{n,m}](z) = Az^{n+1} + \cdots$$

and

$$z^{\lambda_{n+1}}[q_{n+1,m}\mathfrak{f} - p_{n+1,m}](z) = A_2 z^{n+2} + \cdots .$$

Multiplying the first equation by $z^{\lambda_{n+1}}q_{n+1,m}$, the second by $z^{\lambda_n}q_{n,m}$ and deleting one of the equations thus obtained from the other, it follows that

$$z^{\lambda_n+\lambda_{n+1}}[q_{n,m}p_{n+1,m} - q_{n+1,m}p_{n,m}](z) = A_3 z^{n+1} + \cdots$$

Taking into consideration (c.3) we see that on the left hand side we have a polynomial of degree $\leq n+1+m-m^*$. Consequently,

$$z^{\lambda_n+\lambda_{n+1}}[q_{n,m}p_{n+1,m} - q_{n+1,m}p_{n,m}](z) = \tilde{q}_{n,m-m^*}z^{n+1},$$

where $\deg(\tilde{q}_{n,m-m^*}) \leq m - m^*$. Dividing by $z^{\lambda_n+\lambda_{n+1}}q_{n,m}q_{n+1,m}$ and normalizing $\tilde{q}_{n,m-m^*}$ as in (1.20) the desired result is obtained. ∎

The next lemma is very useful for studying the convergence of sequences of rational functions with free poles. It is due to A. A. Gonchar [16, Lemma 1]. For completeness we include the proof. Under appropriate assumptions, it allows to deduce uniform convergence from the weaker σ_1 convergence.

***Lemma* 2.1.1.** *Assume that $\sigma_1 - \lim_n \varphi_n = \varphi,\ K \subset \Omega$, where Ω is a region of the complex plane.*

(i) If all $\varphi_n \in \mathcal{H}(\Omega)$ (the class of analytic functions in Ω), then

$$\lim_n \varphi_n = \varphi, \quad K \subset \Omega,$$

uniformly on compact subsets of Ω and $\varphi \in H(\Omega)$ (more precisely, φ differs from a certain $\varphi_0 \in H(\Omega)$ at most on a set e such that $\sigma_1(e) = 0$).

(ii) If for all $n \in \mathbb{N}$, $\varphi_n \in \mathcal{M}_\mu(\Omega)$ (the class of all meromorphic functions in Ω with at most μ poles counting multiplicities), then $\varphi \in \mathcal{M}_\mu(\Omega)$.

(iii) If for all $n \in \mathbb{N}$, $\varphi_n \in \mathcal{M}(\Omega)$ and φ has exactly μ poles in Ω, then there exists $n_0 \in \mathbb{N}$ such that for $n \geq n_0$ each φ_n has exactly μ poles in Ω. Moreover, if $\zeta \in \Omega$ is a pole of φ of order ν, then for each $\epsilon > 0$ sufficiently small there exists $n_0(\zeta)$ such that for all $n \geq n_0(\zeta)$ the functions φ_n have exactly ν poles in the disk $\{z : \ |z - \zeta| < \epsilon\}$. We express this saying that the poles of φ_n converge as $n \longrightarrow \infty$ to the poles of φ in Ω according to their order. Finally,

$$\lim_n \varphi_n = \varphi, \quad K \subset \Omega',$$

where Ω' is the region obtained deleting from Ω the poles of φ.

Proof. Let $B = \{z : \ |z - z_0| < r\}$ be an arbitrary disk such that $\overline{B} \subset \Omega$. Let $B_1 = \{z : \ |z - z_0| < r_1\}$, $r_1 > r$, be such that $\overline{B_1} \subset \Omega$. It is sufficient to prove that $\lim_n \varphi_n = \varphi$ uniformly on $\overline{B}$. Set $d = (r_1 - r)/4 (> 0)$. Fix $\epsilon > 0$. By hypothesis, there exists n_0 such that for $n \geq n_0$

$$|(\varphi - \varphi_n)(z)| < \frac{\epsilon}{2}, \quad z \in B_1 \setminus e_n, \quad \sigma_1(e_n) < d.$$

Therefore,

$$|(\varphi_n - \varphi_m)(z)| < \epsilon, \quad z \in B_1 \setminus (e_n \cup e_m) \tag{2.1}$$

and

$$\sigma_1(e_n \cup e_m) \leq \sigma_1(e_n) + \sigma_1(e_m) < 2d, \quad n, m \geq n_0.$$

Let us show that for each $n, m \geq n_0$ there exists $r_{n,m}$, $r < r_{n,m} < r_1$, such that
$$C_{n,m} = \{z : \ |z - z_0| = r_{n,m}\} \subset B_1 \setminus (\overline{B} \cup e_n \cup e_m).$$
It fact, let l_{z_0} be a half line departing from z_0. By $(e_n \cup e_m)^*$ we denote the circular projection with center at z_0 of the points in $e_n \cup e_m$ onto l_{z_0}. Then

$$\sigma_1((e_n \cup e_m)^*) \leq \sigma_1(e_n \cup e_m) < 2d = \sigma_1(l_{z_0} \cap (\overline{B_1} \setminus B)).$$

In this chain of relations it is used that under circular proyection the σ_1 content of a set diminishes and that the σ_1 content of a segment is half its length. Therefore, $l_{z_0} \cap (\overline{B_1} \setminus B)$ must contain a point $z_{n,m}$ not belonging to $(e_n \cup e_m)^*$.
Obviously,

$$C_{n,m} = \{z : \ |z - z_0| = |z_{n,m} - z_0|\} \subset B_1 \setminus (\overline{B} \cup e_n \cup e_m)$$

as desired. From (2.1) and the maximum principle

$$|\varphi_n(z) - \varphi_m(z)| \leq \epsilon, \quad z \in \overline{B}.$$

Therefore $\{\varphi_n\}$ satisfies the Cauchy condition for uniform convergence on $\overline{B}$ as we needed to prove.

Fix a bounded region Ω_1 such that $\overline{\Omega_1} \subset \Omega$. Let $\zeta_{n,1}, \cdots \zeta_{n,k}$, $k = k_n \leq \mu$ be the poles of φ_n in $\overline{\Omega_1}$. Set

$$g_n(z) = \prod_{i=1}^{k} (z - \zeta_{n,i})$$

($g_n \equiv 1$ if $k = 0$). Since $\mu < +\infty$, there exists a sequence of indices $\Lambda \subset \mathbb{N}$ such that
$$\lim_{n \in \Lambda} g_n(z) = g(z), \quad \mathcal{K} \subset \mathbb{C},$$
where g is a polynomial not identically equal to zero. Therefore,

$$\sigma_1 - \lim_{n \in \Lambda} g_n \varphi_n = g\varphi, \quad \mathcal{K} \subset \Omega_1.$$

From the first part it follows that $g\varphi \in H(\Omega_1)$ and, hence, $\varphi \in \mathcal{M}_\mu(\Omega_1)$ (because g can have at most μ zeros). Since Ω_1 is arbitrary, we have that $\varphi \in \mathcal{M}_\mu(\Omega_1)$ as claimed.

In order to prove iii) consider a neighborhood B which contains only one of the poles of φ in Ω. If this is a pole of order ν then for all sufficiently large n, φ_n must have at least ν poles in B. Since this is true on a neighborhood of each one of the poles of φ in Ω the statement readily follows. ■

In the study of singular points on the boundary of the convergence region of Taylor and Dirichlet series an important instrument is what is called a regularizing sequence of the sequence of its coefficients. The proof of the following theorem may be found in [1] and [28].

***Theorem* 2.1.1.** *Let $(\alpha_n)_{n\geq 1}$ be a sequence of complex numbers such that*

$$\limsup_{n\to\infty} |\alpha_n|^{1/n} = 1.$$

Then, there exists a sequence $(\alpha_n^*)_{n\geq 1}$ of positive numbers which satisfies:

(i) $\lim\limits_{n\to\infty} \frac{\alpha_n^*}{\alpha_{n+1}^*} = 1,$

(ii) $(log(\frac{\alpha_n^*}{n!}))_{n\geq m}$ *is concave,*

(iii) $|\alpha_n| \leq |\alpha_n^*|,\ n \in \mathbb{Z}_+,$

(iv) $|\alpha_n| \geq c|\alpha_n^*|,\ n \in \Lambda \subset \mathbb{Z}_+,\ c > 0$ *for an infinite sequence Λ of indices.*

The sequence $(\alpha_n^*)_{n\geq 1}$ is called a **regularization** of $(\alpha_n)_{n\geq 1}$. In [34], S.P.Suetin extended the use of regularizing sequences to Padé approximation in order to prove Theorem 1.1.6. His arguments were based on two lemmas [34, Lemmas 1, 2] which we will generalize in order to adjust them for the study of singularities of incomplete Padé approximation. The next section is dedicated to their proofs.

§ 2.2. Two fundamental lemmas.

The first lemma concerns bounds related with incomplete Padé approximants on compact subsets of the complement of the circle $\{z : |z| = R_m^*\}$ defining D_m^*, see (1.23). We will assume that $0 < R_m^* < +\infty$. In this case, making a change of variables if necessary, we can assume that $R_m^* = 1$.

***Lemma* 2.2.1.** *Let $\mathfrak{f}$ be a formal power series. Fix $m \geq m^* \geq 1$ and assume that*

$$\limsup_{n\to\infty} |A_{n,m}|^{1/n} = 1,$$

where the coefficients $A_{n,m}$ are the ones appearing in (1.22). *Let $\left(A_{n,m}^*\right)_{n\geq m}$ be a regularizing sequence associated with $\left(A_{n,m}\right)_{n\geq m}$. Then*

1. *for any $\delta > 0$*

$$\max_{|z|\geq e^\delta} \left| \frac{p_{n,m}(z)}{A_{n,m}^* z^n} \right| = \mathcal{O}(1), \qquad n \to \infty \tag{2.2}$$

2. for every compact $K \subset \{z : |z| < e^{-\delta}\} \setminus \mathcal{P}(\mathfrak{f})$, where $\mathcal{P}(\mathfrak{f})$ is the set of poles of $\mathfrak{f}$,

$$\max_{z \in K} \left| \frac{(q_{n,m}\mathfrak{f} - p_{n,m})(z)}{A^*_{n,m} z^n} \right| = \mathcal{O}(1), \qquad n \to \infty. \tag{2.3}$$

Proof. Let $r_{n,m} = \frac{p_{n,m}}{q_{n,m}}, n = 1, 2, ...,$ where the polynomials $p_{n,m}$ and $q_{n,m}$ do not have common zeros.The set $\mathcal{P}_{n,m}(\mathfrak{f}) = \{\zeta_{n,1}, ..., \zeta_{n,m_n}\}$ denotes the set of zeros of $q_{n,m}$.

Now consider the difference

$$(r_{n,m} - r_{m,m})(z) = \sum_{k=m}^{n-1} \frac{A_{k,m} z^{k+1-\lambda_k-\lambda_{k+1}} q^*_{k,m-m^*}(z)}{(q_{k,m}q_{k+1,m})(z)} =$$

$$A^*_{n,m} z^n \sum_{k=m}^{n-1} \frac{A_{k,m} z^{k+1-\lambda_k-\lambda_{k+1}} q^*_{k,m-m^*}(z)}{A^*_{n,m} z^n (q_{k,m}q_{k+1,m})(z)} =$$

$$A^*_{n,m} z^n \sum_{k=m}^{n-1} \frac{A_{k,m}}{A^*_{n,m}} z^{k-n} \frac{z^{1-\lambda_k-\lambda_{k+1}} q^*_{k,m-m^*}(z)}{(q_{k,m}q_{k+1,m})(z)}.$$

Therefore

$$|(r_{n,m} - r_{m,m})(z)| = \left| A^*_{n,m} z^n \sum_{k=m}^{n-1} \frac{A_{k,m}}{A^*_{n,m}} z^{k-n} \frac{z^{1-\lambda_k-\lambda_{k+1}} q^*_{k,m-m^*}(z)}{(q_{k,m}q_{k+1,m})(z)} \right| \leq$$

$$\left| A^*_{n,m} z^n \right| \sum_{k=m}^{n-1} \left| \frac{A_{k,m}}{A^*_{n,m}} \right| |z|^{k-n} \frac{|z|^{1-\lambda_k-\lambda_{k+1}} \left| q^*_{k,m-m^*}(z) \right|}{|(q_{k,m}q_{k+1,m})(z)|}.$$

By (iii) of Theorem 2.1.1 we have

$$|(r_{n,m} - r_{m,m})(z)| \leq \left| A^*_{n,m} z^n \right| \sum_{k=m}^{n-1} \left| \frac{A^*_{k,m}}{A^*_{n,m}} \right| |z|^{k-n} \frac{|z|^{1-\lambda_k-\lambda_{k+1}} \left| q^*_{k,m-m^*}(z) \right|}{|(q_{k,m}q_{k+1,m})(z)|}$$

where $z \in \mathcal{K}$.

The property (ii) of Theorem 2.1.1 implies that

$$\left| A^*_{n-1,m} A^*_{n+1,m} \right| \leq (A^*_{n,m})^2.$$

Then

$$\left| \frac{A^*_{n-1,m}}{A^*_{n,m}} \right| \leq \left| \frac{A^*_{n,m}}{A^*_{n+1,m}} \right|.$$

Therefore, $\left\{\frac{A^*_{k,m}}{A^*_{k+1,m}}\right\}$ monotonically increases to 1 due to (i) and

$$\frac{A^*_{k,m}}{A^*_{k+1,m}} \leq 1.$$

This implies

$$\left|\frac{A^*_{k,m}}{A^*_{n,m}}\right| = \left|\frac{A^*_{k,m}}{A^*_{k+1,m}}\right| \left|\frac{A^*_{k+1,m}}{A^*_{k+2,m}}\right| \cdots \left|\frac{A^*_{n-1,m}}{A^*_{n,m}}\right| \leq 1.$$

Hence

$$|(r_{n,m} - r_{m,m})(z)| \leq \left|A^*_{n,m} z^n\right| \sum_{k=m}^{n-1} |z|^{k-n} \frac{|z|^{1-\lambda_k-\lambda_{k+1}} \left|q^*_{k,m-m^*}(z)\right|}{|(q_{k,m}q_{k+1,m})(z)|}.$$

Fix a compact set $\mathcal{K} \subset \{z : \ |z| > 1\}$ and let $z' \in \mathcal{K}$. Set $U_{2r}(z') = \{z : |z - z'| < 2r\}$. Take r sufficiently small so that $|z| > 1$ for all $z \in \overline{U_{2r}(z')}$. Then $|z| \geq \frac{1}{\alpha}$, $0 < \alpha < 1$ for all $z \in U_{2r}(z')$. Consequently,

$$|(r_{n,m} - r_{m,m})(z)| \leq |A^*_{n,m} z^n| \sum_{k=m}^{n-1} \alpha^{n-k} \frac{|z|^{1-\lambda_n-\lambda_{n+1}} |q^*_{k,m-m^*}(z)|}{|(q_{k,m}q_{k+1,m})(z)|}.$$

or, what is the same,

$$|(r_{n,m} - r_{m,m})(z)| \leq C_1 |A^*_{n,m} z^n| \sum_{k=1}^{n-m} \alpha^k \frac{|q^*_{n-k,m-m^*}(z)|}{|(q_{n-k,m}q_{n-k+1,m})(z)|} \tag{2.4}$$

Since $q^*_{k,m-m^*}(z)$ is normalized as in (1.20) we have that

$$\left\|q^*_{k,m-m^*}\right\|_{\mathcal{K}} = \max_{z \in \mathcal{K}} |q^*_{k,m-m^*}(z)| \leq C < +\infty$$

where C does not depend on $k \in \mathbb{N}$. Obviously, $\deg(q_{n-k,m}q_{n-k+1,m}) \leq 2m$, $k = 1, 2, ..., n-m$. Take $\epsilon > 0$ so that

$$\epsilon \sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{2r}{3} < r$$

For each $k = 1, 2, ..., n-m$ let $V_{k,\epsilon}$ be the set consisting of the $(\epsilon/(4mk^2))-$ neighborhood of the zeros of the polynomial $(q_{n-k,m}q_{n-k+1,m})$ and let $V_n^\epsilon = \bigcup_{k=1}^{n-m} V_{k,\epsilon}$. The sum of the diameters of the disks in V_n^ϵ does not exceed

$\epsilon \sum_{k=1}^{\infty} \frac{1}{k^2} < r$. Therefore, there is a circle γ_n centered at z' of radius r_n, $r < r_n < 2r$ which does not intersect V_n^{ϵ}. Then, for all $z \in \gamma_n$ and $k = 1, 2, ..., n-1$

$$|(q_{n-k,m}q_{n-k+1,m})(z)| \geq C_2(\epsilon/4mk^2)^{2m}$$

where $C_2 > 0$ does not depend on k. Consequently, from (2.4)

$$|(r_{n,m} - r_{m,m})(z)| \leq C_3|A^*_{n,m}z^n|(4m/\epsilon)^{2m}\sum_{k=1}^{n-1}\alpha^k k^{4m} \leq C_4|A^*_{n,m}z^n|, \quad (2.5)$$

since the series $\sum_{k=1}^{\infty} \alpha^k k^{4m}$ converges because $0 < \alpha < 1$. Now $|A^*_{n,m}z^n| \rightrightarrows \infty$ as $n \longrightarrow \infty$ in $U_{2r}(z')$; therefore, (2.5) implies the inequality

$$|r_{n,m}(z)| \leq C_5|A^*_{n,m}z^n|, \quad z \in \gamma_n, \quad n \in \Lambda. \quad (2.6)$$

Multiplying both sides of (2.6) by $q_{n,m}$, using (1.21), and the maximum modulus principle for holomorphic functions, we get the estimate

$$\left|\frac{p_{n,m}(z)}{A^*_{n,m}z^n}\right| \leq C_6, \qquad z \in \overline{U_{2r}(z')}, \quad n \in \Lambda. \quad (2.7)$$

By the Heine-Borel Theorem it follows that (2.7) is true for all $z \in \mathcal{K}$. Then the desired result (2.2) follows immediately taking $\mathcal{K} = \{z : \ |z| = e^{\delta}\}$, $\delta > 0$, using the maximum principle and the fact that $p_{n,m}/A^*_{n,m}z^n$ is holomorphic in $\{z : |z| > 1\} \cup \{\infty\}$.

Fix a compact subset $\mathcal{K}$ contained in $\{z : \ |z| < 1\} \setminus \mathcal{P}(\mathfrak{f})$ and $z \in \mathcal{K}$. Choose $r > 0$ sufficiently small so that $\overline{U_{2r}(z')} \subset \{z : \ |z| < 1\} \setminus \mathcal{P}(\mathfrak{f})$. By the $\sigma_1 -$ *convergence* of the sequence $(r_{n,m})_{n\geq m}$ to $\mathfrak{f}$ on compact subsets of $\{z : |z| < 1\}$, the next representation holds for almost all circles centered at z' contained in $U_{2r}(z')$

$$\mathfrak{f}(z) = r_{n,m}(z) + \sum_{k=n}^{\infty} \frac{A_{k,m}z^{k+1-\lambda_k-\lambda_{k+1}}q^*_{k,m-m^*}(z)}{(q_{k,m}q_{k+1,m})(z)}. \quad (2.8)$$

That is

$$(\mathfrak{f} - r_{n,m})(z) = A^*_{n,m}z^n \sum_{k=n}^{\infty} \frac{A_{k,m}}{A^*_{n,m}} z^{k-n} \frac{z^{1-\lambda_k-\lambda_{k+1}}q^*_{k,m-m^*}(z)}{(q_{k,m}q_{k+1,m})(z)}.$$

Then, on any such circle

$$|(\mathfrak{f} - r_{n,m})(z)| \le |A^*_{n,m} z^n| \sum_{k=n}^{\infty} \left| \frac{A_{k,m}}{A^*_{n,m}} \right| |z|^{k-n} \frac{|z|^{1-\lambda_k - \lambda_{k+1}} |q^*_{k,m-m^*}(z)|}{|(q_{k,m} q_{k+1,m})(z)|}.$$

By (iii) of Theorem 2.1.1 we have

$$|(\mathfrak{f} - r_{n,m})(z)| \le |A^*_{n,m} z^n| \sum_{k=n}^{\infty} \left| \frac{A^*_{k,m}}{A^*_{n,m}} \right| |z|^{k-n} \frac{|z|^{1-\lambda_k - \lambda_{k+1}} |q^*_{k,m-m^*}(z)|}{|(q_{k,m} q_{k+1,m})(z)|}.$$

where $z \in \mathcal{K}$.

We know that

$$\left| \frac{A^*_{k,m}}{A^*_{n,m}} \right| = \left| \frac{A^*_{k,m}}{A^*_{k-1,m}} \right| \left| \frac{A^*_{k-1,m}}{A^*_{k-2,m}} \right| \cdots \left| \frac{A^*_{n+1,m}}{A^*_{n,m}} \right| \le \left| \frac{A^*_{n+1,m}}{A^*_{n,m}} \right|^{k-n}$$

On account of property (i) in Theorem 2.1.1, for any $\epsilon > 0$ there exists n_0 such that if $n \ge n_0$

$$\left| \frac{A^*_{n+1,m}}{A^*_{n,m}} \right| < (1+\epsilon).$$

Take $\epsilon > 0$ sufficiently small, such that

$$|1+\epsilon|\,|z| \le \alpha < 1, \qquad z \in \overline{U_{2r}(z')}.$$

Using (1.21) it follows that

$$|(\mathfrak{f} - r_{n,m})(z)| \le C |A^*_{n,m} z^n| \sum_{k=n}^{\infty} \alpha^{k-n} \frac{|z|^{1-\lambda_k - \lambda_{k+1}}}{|(q_{k,m} q_{k+1,m})(z)|}. \tag{2.9}$$

on almost any circle centered at z' contained in $\overline{U_{2r}(z')}$.

Now, define $\widehat{V}_{k,\epsilon}$ as the set consisting of the $(\epsilon/4m(k+1-n)^2)$ −neighborhood of the zeros of the polynomial $q_{k,m} q_{k+1,m}$, $k \ge n$, and $\widehat{V}^{\epsilon}_n = \bigcup_{k=n}^{\infty} \widehat{V}_{k,\epsilon}$. Take $\epsilon > 0$ so that

$$\epsilon \sum_{k=1}^{\infty} \frac{1}{k^2} \le \frac{2r}{3} < r.$$

The sum of the diameters of the disks constituting $\widehat{V}^{\epsilon}_n$ does not exceed $\epsilon \sum_{k=1}^{\infty} \frac{1}{k^2} < r$. Therefore, there is a circle γ_n, $0 \notin \gamma_n$, centered at z' of radius r_n, $r < r_n < 2r$, which does not intersect $\widehat{V}^{\epsilon}_n$. Then, for all $z \in \gamma_n$ and

$k \geq n$.

$$|(q_{k,m}q_{k+1,m})(z)| \geq C_1 \left(\frac{\epsilon}{4m(k+1-n)^2} \right)^{2m}$$

and using (2.9) we obtain

$$|(\mathfrak{f} - r_{n,m})(z)| \leq C_1 |A^*_{n,m} z^n| \sum_{k=n}^{\infty} \alpha^{k-n}(k+1-n)^{4m} \leq C_3 \left| A^*_{n,m} z^n \right| \qquad (2.10)$$

since $\sum\limits_{k=0}^{\infty} \alpha^k (k+1)^{4m} < +\infty$.

From (2.10) it follows that

$$\left| \frac{(q_{n,m}\mathfrak{f} - p_{n,m})(z)}{A^*_{n,m} z^n} \right| \leq C_4, \qquad z \in \gamma_n,$$

and from the maximum principle, we obtain

$$\left| \frac{(q_{n,m}\mathfrak{f} - p_{n,m})(z)}{A^*_{n,m} z^n} \right| \leq C_5, \qquad z \in U_r(z').$$

Using the Heine-Borel Theorem it follows that

$$\left| \frac{(q_{n,m}\mathfrak{f} - p_{n,m})(z)}{A^*_{n,m} z^n} \right| \leq C_6, \quad z \in \mathcal{K}.$$

Now $\mathcal{K} \subset \{z : \quad |z| < 1\} \setminus \mathcal{P}(\mathfrak{f})$; therefore, (2.3) follows immediately. ∎

The second lemma refers to bounds on neighborhoods of arcs contained in $\{z : |z| = R^*_m\}$ so the proof is more complicated. We will assume that the denominators of the incomplete Padé approximants converge. As before, we restrict our attention to the case when R^*_m is finite and without loss of generality consider that $R^*_m = 1$.

***Lemma* 2.2.2.** *Let $\mathfrak{f}$ be a formal power series. Assume that* $\limsup_n |A_{n,m}|^{1/n} = 1$ *and* $\lim_{n,m} q_{n,m} = q$ *where q is a polynomial of degree m. Suppose that $\mathfrak{f}$ is holomorphic at the point z_0, $|z_0| = 1$. Then there is a $\delta = \delta(z_0) > 0$ such that*

$$\max_{e^{-\delta} \leq |z| \leq e^{\delta}, |\arg(z) - \arg(z_0)| \leq \delta} \left| \frac{(q_{n,m}\mathfrak{f} - p_{n,m})(z)}{A^*_{n,m} z^n} \right| = \mathcal{O}(1), \quad n \to \infty \qquad (2.11)$$

where $\arg(z)$ *denotes the argument of the complex number z.*

Before proving Lemma 2.2.2 we introduce some tools to be used. For this we single out some inequalities connected with the function e^z and present the basic properties of the Borel transform that we shall need later. Let $S_n(z)$ be the nth partial sum of the Taylor series of e^z with center at $z = 0$.

We know that

$$S_{n-1}(z) = \sum_{k=0}^{n-1} \frac{z^k}{k!}$$

and

$$e^z = \sum_{k=0}^{\infty} \frac{z^k}{k!}.$$

Then

$$e^z - S_{n-1}(z) = \sum_{k=n}^{\infty} \frac{z^k}{k!}.$$

Dividing by $\frac{1}{n!}z^n$ and applying the triangle inequality we have

$$\frac{\left|\sum\limits_{k=n}^{\infty} \frac{z^k}{k!}\right|}{\left|\frac{1}{n!}z^n\right|} \leq 1 + \frac{|z|}{n+1} + \frac{|z|^2}{(n+2)(n+1)} + \cdots \leq$$

$$1 + \frac{|z|}{n} + \frac{|z|^2}{n^2} + \frac{|z|^3}{n^3} + \cdots =$$

$$\sum_{k=0}^{\infty} \left|\frac{z}{n}\right|^k = \frac{1}{1 - \left|\frac{z}{n}\right|}, \qquad |z| < |n|.$$

Thus

$$\left|\frac{e^z - S_{n-1}(z)}{z^n/n!}\right| \leq \frac{1}{1 - |z/n|}, \qquad |z| < n. \tag{2.12}$$

Similarly, we obtain

$$\left|\frac{S_{n-1}(z)}{z^n/n!}\right| \leq \frac{1}{|z/n| - 1}, \qquad |z| > n. \tag{2.13}$$

Hence

$$\max_{|z| \leq ne^{-\delta}} \left|\frac{e^z - S_{n-1}(z)}{(1/n!)z^n}\right| = \mathcal{O}(1), \qquad n \to \infty \tag{2.14}$$

and

$$\max_{|z| \geq ne^{\delta}} \left|\frac{S_{n-1}(z)}{(1/n!)z^n}\right| = \mathcal{O}(1), \qquad n \to \infty \tag{2.15}$$

Moreover, we have

$$c(\delta)\left|\frac{z^k}{k!}\right| \le \left|\frac{z^{k+m}}{(k+m)!}\right| \le C(\delta)\left|\frac{z^k}{k!}\right|, \tag{2.16}$$

where $ne^{-\delta} \le |z| \le ne^{\delta}$, $ne^{-\delta} \le k \le k+m \le ne^{\delta}$, and $C(\delta)$ and $c(\delta)$ depend only on δ and m.

On the ray $\arg(z) = \varphi$ we have $|e^z| = e^{|z|\cos\varphi}$. Therefore, using Stirling's formula it is easy to deduce that for every $\epsilon > 0$ there is a $\delta > 0$ such that

$$\max_{ne^{-\delta}\le|z|\le ne^{\delta}, |\arg(z)|\ge\epsilon}\left|\frac{e^z}{z^n/n!}\right| = o(1), \qquad n\to\infty. \tag{2.17}$$

Let us prove the following bound.

***Proposition* 2.2.1.** *For every $\epsilon > 0$ there exists $\delta > 0$ such that*

$$\max_{|z|\le ne^{\delta}, |\arg(z)|\ge\epsilon}\left|\frac{e^z - S_{n-1}(z)}{z^n/n!}\right| = \mathcal{O}(1), \qquad n\to\infty. \tag{2.18}$$

Proof. Fix $\delta > 0, \epsilon > 0$. In order to prove (2.18), due to (2.14) it rests to show that

$$\max_{ne^{-\delta}\le|z|\le ne^{\delta},\ |\arg(z)|\ge\epsilon}\left|\frac{e^z - S_{n-1}(z)}{z^n/n!}\right| = \mathcal{O}(1), \qquad n\to\infty. \tag{2.19}$$

Consider the closed curve

$$\gamma_n = \gamma_{n,1}\cup\gamma_{n,2}\cup\gamma_{n,3}\cup\gamma_{n,4},$$

where

$$\gamma_{n,1} = \{z : |z| = ne^{-2\delta}, |\arg z| \ge \epsilon/2\}, \quad \gamma_{n,2} = \{z : |z| = ne^{2\delta}, |\arg z| \ge \epsilon/2\},$$
$$\gamma_{n,3} = \{z : z = re^{i\epsilon/2}, ne^{-2\delta} \le r \le ne^{2\delta}\},$$
$$\gamma_{n,4} = \{z : z = re^{-i\epsilon/2}, ne^{-2\delta} \le r \le ne^{2\delta}\},$$

The proof consists in obtaining adequate bounds on the curves $\gamma_{n,k}, k = 1,\cdots,4$ of an auxiliary function.

Notice that for $|z| \le ne^{2\delta}$

$$\left|1 - \frac{z}{ne^{\pm i\epsilon/2}}\right| \le (1 + e^{2\delta}). \tag{2.20}$$

Consider the auxiliary function

$$\varphi_n(z) := \left(1 - \frac{z}{ne^{i\epsilon/2}}\right)\left(1 - \frac{z}{ne^{-i\epsilon/2}}\right)\frac{e^z - S_{n-1}(z)}{z^n/n!}.$$

From (2.14) and (2.20), we have

$$\max_{z\in\gamma_{n,1}} |\varphi_n(z)| = \mathcal{O}(1), \qquad n \to \infty$$

The same bound is obtained on $\gamma_{n,2}$ as a consequence of (2.15), (2.17), and (2.20).

The bounds on $\gamma_{n,3}$ and $\gamma_{n,4}$ are obtained similarly, so we will restrict our attention on $\gamma_{n,3}$. Let us divide this curve in two. We denote by $\gamma_{n,3,+}$ those points of $\gamma_{n,3}$ whose absolute value is smaller than n and $\gamma_{n,3,-} = \gamma_{n,3} \setminus \gamma_{n,3,+}$. Without loss of generality we can assume that $0 < \epsilon < \pi/2$.

Using (2.12), for $z \in \gamma_{n,3,+}$ we have

$$\max_{\gamma_{n,3,+}} |\varphi_n(z)| \le \max_{\gamma_{n,3,+}} \frac{\left|\left(1 - \frac{z}{ne^{i\epsilon/2}}\right)\left(1 - \frac{z}{ne^{-i\epsilon/2}}\right)\right|}{1 - |z/n|} \le \max_{r\le n} \left|1 - \frac{re^{2i\epsilon}}{n}\right| \le 2.$$

On the other hand, from (2.13), (2.17), and (2.20)

$$\begin{aligned}
\max_{\gamma_{n,3,-}\setminus\{ne^{i\epsilon/2}\}} |\varphi_n(z)| &\le \max_{\gamma_{n,3,-}\setminus\{ne^{i\epsilon/2}\}} \left|\left(1 - \frac{z}{ne^{i\epsilon/2}}\right)\left(1 - \frac{z}{ne^{-i\epsilon/2}}\right)\frac{e^z}{z^n/n!}\right| + \\
&\max_{\gamma_{n,3,-}\setminus\{ne^{i\epsilon/2}\}} \left|\left(1 - \frac{z}{ne^{i\epsilon/2}}\right)\left(1 - \frac{z}{ne^{-i\epsilon/2}}\right)\frac{S_{n-1}(z)}{z^n/n!}\right| \le \\
&o(1) + \max_{\gamma_{n,3,-}\setminus\{ne^{i\epsilon/2}\}} \frac{\left|\left(1 - \frac{z}{ne^{i\epsilon/2}}\right)\left(1 - \frac{z}{ne^{-i\epsilon/2}}\right)\right|}{|z/n| - 1} \le \\
&o(1) + \max_{r\le ne^{2\delta}} \left|1 - \frac{re^{2i\epsilon}}{n}\right| \le o(1) + 1 + e^{2\delta} = \mathcal{O}(1), \qquad n \to \infty.
\end{aligned}$$

Putting these estimates together, it follows that

$$\max_{\gamma_{n,3}} |\varphi_n(z)| = \mathcal{O}(1), \qquad n \to \infty$$

(At the point $z = ne^{i\epsilon/2}$ the bound is true by continuity.)

We have proved that

$$\max_{\gamma_n} |\varphi_n(z)| = \mathcal{O}(1), \qquad n \to \infty.$$

Since γ_n surrounds the set on which we wish to prove (2.19), by the maximum principle

$$\max_{ne^{-\delta}\le|z|\le ne^{\delta},\ |\arg(z)|\ge\epsilon} |\varphi_n(z)| = \mathcal{O}(1), \qquad n \to \infty.$$

Now

$$\min_{|z|\leq ne^{\delta},\ |\arg(z)|\geq\epsilon}\left|\left(1-\frac{z}{ne^{i\epsilon/2}}\right)\left(1-\frac{z}{ne^{-i\epsilon/2}}\right)\right|\geq$$

$$\min_{|\arg(z)|\geq\epsilon}\left|\left(1-ze^{i\epsilon/2}\right)\left(1-ze^{-i\epsilon/2}\right)\right|\geq\sin^2(\epsilon/2)>0.$$

Consequently, (2.19) follows and we are done. ■

Let the following conditions be satisfied: for all sufficiently large n the function $r_{n,m}$ has exactly m finite poles $\xi_{n,1},\xi_{n,2},\cdots,\xi_{n,m}$ and

$$\xi_{n,j}\to a_j\in\mathbb{C}\setminus\{0\},\qquad \text{as}\qquad n\to\infty,\quad j=1,\cdots,m. \tag{2.21}$$

That is to say

$$\lim_{n\to\infty}q_{n,m}(z)=q_m(z)=C\prod_{j=1}^{m}(z-a_j),$$

where C is a constant. Recall that $q_{n,m}$ is the denominator of $r_{n,m}$ after canceling out all common factors with the numerator. Therefore, (2.21) implies that $\deg q_{n,m}=m$ for all sufficiently large n. If follows that $\lambda_n=0$ for all sufficiently large n. It is worth mentioning that (2.21) together with Lemma 2.1.1 and Theorem 1.3.1 imply that $\lim_{n\to\infty}r_{n,m}=\mathfrak{f}$, uniformly on each compact subset of $D_m^*(\mathfrak{f})\setminus\{a_1,\ldots,a_m\}$.

On the other hand, taking (2.21) into consideration we can normalize the polynomials $q_{n,m}$ to be monic so that $C=1$. The passage from one normalization (see (1.20)) to this new one means that the initial coefficients $A_{n,m}$ are multiplied by certain numbers (depending on the zeros of the polynomials $q_{n,m}$ that have for limit a quantity that is different from zero, since $a_j\neq 0, j=1,\ldots,n$. Therefore, $\limsup_{n\to\infty}|A_{n,m}|^{1/n}$ does not change its value after the renormalization.

To summarize, in the rest of this section we suppose that (2.21) takes place, assume that the polynomials $q_{n,m}$ are monic and $\lambda_n=0$ for all sufficiently large n.

As before, $1/R_m^*=\limsup_{n\to\infty}|A_{n,m}|^{1/n}$ and $D_m^*(f)$ denotes the disk of radius R_m^*. Without loss of generality we suppose that $R_m^*=1$.

The proof of Lemma 2.2 is based on the use of the Borel transform $B(g)$ of a holomorphic function g. Let

$$g(z)=\sum_{k=0}^{\infty}g_kz^k$$

be a power series with radius of convergence equal to $R>0$. The Borel transform of g is defined as

$$B(g)(z):=\sum_{k=0}^{\infty}\frac{g_k}{k!}z^k.$$

It readily follows that

$$B(g)(z) = \sum_{k=0}^{\infty} \frac{g_k}{k!} z^k = \frac{1}{2\pi i} \int_{\Gamma} g(t) e^{z/t} \frac{dt}{t}, \tag{2.22}$$

where Γ is any contour lying in the domain of holomorphy of g surrounding the origin. From the integral representation we see that $B(g)$ is an entire function. When $R = 1$ it is known that $B(g)$ is of exponential type ≤ 1 and if g is holomorphic at $z_0 = e^{i\varphi_0}$, there are a number h, $0 < h < 1$, and an $\epsilon > 0$ such that

$$|B(g)(z)| \leq C e^{h|z|} \tag{2.23}$$

for $|\arg z - \varphi_0| \leq \epsilon$, where C is independent of z.

It is well known (see [5]) that g can be recovered from $B(g)$ in neighborhoods of its points of holomorphy on the unit circle by

$$g(re^{i\varphi}) = \frac{1}{r} \int_0^{\infty} B(g)(\rho e^{i\varphi}) e^{-\rho/r} d\rho \tag{2.24}$$

From (2.23) it follows that in a neighborhood of $z_0 = e^{i\varphi_0}$ formula (2.24) is valid for all $r < 1/h$, where $1/h > 1$ and $|\varphi - \varphi_0| \leq \epsilon$.

According to Theorem 1.3.1, D_m^* is the largest disk inside of which $\sigma_1 - \lim_{n\to\infty} r_{n,m} = \mathfrak{f}$. It follows that each pole of $\mathfrak{f}$ in D_m^* attracts as many zeros of the polynomials $q_{n,m}$ as its multiplicity (see proof of 2.1.1). Therefore, if D_m^* contains at least m^* poles of $\mathfrak{f}$ then q_m has at least m^* zeros that coincide with singularities of $\mathfrak{f}$ (counting multiplicities). We wish to prove that is the case regardless of the number of poles which $\mathfrak{f}$ has in D_m^*. Since the assertion is true when f has at least m^* poles in D_m^* we can restrict our attention to the case when the number of poles of $\mathfrak{f}$ in D_m^* is less than m^*. Let $a_1, \ldots, a_\mu, \mu < m^*$, be the zeros of q_m inside D_m^* which are poles of $\mathfrak{f}$ counting multiplicities and set

$$\omega(z) = \prod_{j=1}^{\mu} (z - a_j).$$

By (2.21) the coefficients of the polynomials $q_{n,m} = z^m + \cdots$ are bounded as $n \to \infty$. Hence by (2.23), replacing g by the functions $z^j (f\omega)(z)$, $j = 0, 1, \cdots, m$, it follows that there are numbers h, $0 < h < 1$, and $\epsilon' > 0$ independent of n such that, when $|\arg(z)| \leq \epsilon'$,

$$|B(q_{n,m} f\omega)(z)| \leq C_1 e^{h|z|}. \tag{2.25}$$

In the sequel, $C_1, C_2, \cdots$ denote positive quantities, independent of n and z, and also of the summation index k and the variable t used below.

If we define

$$F_n(z) = (q_{n,m}\mathfrak{f}\omega - p_{n,m}\omega)(z) \tag{2.26}$$

Then in accordance with (2.24) and (2.25) we have

$$F_n(re^{i\varphi}) = \frac{1}{r}\int_0^\infty B(F_n)(\rho e^{i\varphi})e^{-\rho/r}d\rho, \tag{2.27}$$

where $r < h^{-1}, |\varphi| \leq \epsilon'$, and h and ϵ' are independent of n. Note also that

$$\frac{F_n(z)}{z^{n+1}} = \frac{(q_{n,m}\mathfrak{f}\omega - p_{n,m}\omega)(z)}{z^{n+1}} \tag{2.28}$$

is holomorphic in D_m^* since $(q_{n,m}\mathfrak{f} - p_{n,m})(z) = \mathcal{O}(z^{n+1})$ when $z \to 0$, and ω eliminates the poles of $\mathfrak{f}$ in this disk.

***Proposition* 2.2.2.** *For every $\delta > 0$*

$$\max_{|z|\leq ne^{-\delta}} \left|\frac{B(F_n)(z)}{(A_n^*/n!)z^n}\right| = \mathcal{O}(1), \qquad n \longrightarrow \infty \tag{2.29}$$

Proof. Let $\rho = e^{-\delta/2}$ and $\Gamma_\rho = \{z: \ |z| = \rho\}$. Then by (2.22) and (2.28) we have

$$B(F_n)(z) = \frac{1}{2\pi i}\int_{\Gamma_\rho} F_n(t)e^{z/t}\frac{dt}{t} =$$

$$\frac{1}{2\pi i}\int_{\Gamma_\rho}(q_{n,m}\mathfrak{f}\omega - p_{n,m}\omega)(t)\left[e^{z/t} - S_{n-1}\left(\frac{z}{t}\right)\right]\frac{dt}{t} =$$

$$\frac{A_n^*}{n!}z^n\frac{1}{2\pi i}\int_{\Gamma_\rho}\frac{(q_{n,m}\mathfrak{f}\omega - p_{n,m}\omega)(t)}{A_n^* t^n}\frac{\left[e^{z/t} - S_{n-1}\left(z/t\right)\right]}{(1/n!)(z/t)^n}\frac{dt}{t}$$

Since $|z/t| \leq ne^{-\delta/2}$ for $t \in \Gamma_\rho$ and $|z| \leq ne^{-\delta}$, we obtain (2.29) from (2.3) and (2.14). In the second equality it is used that

$$\frac{1}{2\pi i}\int_{\Gamma_\rho}(q_{n,m}\mathfrak{f}\omega - p_{n,m}\omega)(t)S_{n-1}\left(\frac{z}{t}\right)\frac{dt}{t} = 0$$

which follows from the fact that the function under the integral sign is holomorphic in the unit disk with respect to t. ■

We also have

***Proposition* 2.2.3.** *For every $\delta > 0$*

$$\max_{|z|\geq ne^{\delta}} \left| \frac{B(p_{n,m}\omega)(z)}{(A_n^*/(n)!)z^n} \right| = \mathcal{O}(1), \qquad n \longrightarrow \infty \tag{2.30}$$

Proof. Let $\rho = e^{\delta/2} > 1$ then

$$B(p_{n,m}\omega)(z) = \frac{1}{2\pi i} \int_{\Gamma_\rho} (p_{n,m}\omega)(t) e^{z/t} \frac{dt}{t} =$$

$$\frac{1}{2\pi i} \int_{\Gamma_\rho} (p_{n,m}\omega)(t) S_{n-1}\left(\frac{z}{t}\right) \frac{dt}{t} + \frac{1}{2\pi i} \int_{\Gamma_\rho} (p_{n,m}\omega)(t) \left[e^{z/t} - S_{n-1}\left(\frac{z}{t}\right)\right] \frac{dt}{t}$$

The last integral is zero, since $deg(p_{n,m}\omega) \leq n-1$ and thus the function under the integral sign has a zero of order ≥ 2 at infinity with respect to t. Consequently

$$B(p_{n,m}\omega)(z) = \frac{1}{2\pi i} \int_{\Gamma_\rho} (p_{n,m}\omega)(t) S_{n+m-1}\left(\frac{z}{t}\right) \frac{dt}{t} =$$

$$\frac{A_n^* z^n}{n!} \frac{1}{2\pi i} \int_{\Gamma_\rho} \frac{(p_{n,m}\omega)(t)}{A_n^* t^n} \frac{S_{n-1}(z/t)}{(1/n!)(z/t)^n} \frac{dt}{t}.$$

Since $|t| = e^{\delta/2}$ we have $|z/t| \geq ne^{\delta/2}$ for $|z| \geq ne^{\delta}$. Consequently, (2.30) follows from (2.2) and (2.15). ∎

Suppose that z_0 is a point on the boundary of D_m^* where f is holomorphic (therefore $|z_0| = 1$, according to our assumption on R_m^*). Without loss of generality, making a rotation in the variable if necessary, we may assume that $z_0 = 1$. The function $(\mathfrak{f}\omega)(z)$ is holomorphic in D_m^*, and on a neighborhood of $z = 1$.

***Proposition* 2.2.4.** *If $\mathfrak{f}$ is holomorphic at $z_0 = 1$, there is a $\delta > 0$ such that*

$$\max_{ne^{-\delta}\leq|z|\leq ne^{\delta},\ |\arg z|\leq\delta} \left| \frac{B(F_n)(z)}{(A_n^*/n!)z^n} \right| = \mathcal{O}(1), \qquad n \to \infty \tag{2.31}$$

Proof. Let us consider first the case when $z_0 \neq a_j, j = 1, \ldots, m$. We will show there are numbers $\delta > 0$ and $\epsilon > 0$ such that, for $ne^{-\delta} \leq |z| \leq ne^{\delta}$ and $|\arg z| \leq \epsilon$,

$$|B(F_n)(z)| \leq C_2 \sum_{k=n}^{\infty} \frac{A_k^*}{k!} |z|^k \tag{2.32}$$

and

$$|B(F_n)(z)| \le C_3 \sum_{k=n_0}^{n-1} \frac{A_k^*}{k!}|z|^k, \tag{2.33}$$

where n_0 is sufficiently large. Let ϵ satisfy the following conditions: $0 < \epsilon < \epsilon'/2$, the function $\mathfrak{f}$ is holomorphic on the set

$$\Delta_\epsilon = \left\{z : \ e^{-2\epsilon} \le |z| \le e^{2\epsilon}, |\arg z| \le 2\epsilon\right\},$$

and the points $a_1, a_2, \cdots, a_m$ are outside Δ_ϵ.

Now choose $\delta = \delta(\epsilon, h) \in (0, \epsilon)$ so that (2.18) holds and the following inequalities are satisfied

$$x^x e^{e^{2\delta}\cos\epsilon + 2\delta x - x} < 1 \tag{2.34}$$

for all $x \in [e^{-3\delta}, 1]$,

$$e^{he^{\delta} - 1 + \delta} < 1 \tag{2.35}$$

where the value of h is given in (2.25),

$$e^{h - e^{-\delta}} < 1 \tag{2.36}$$

and $e^{-\delta} > \max\limits_{|a_j|<1} |a_j|$. The reason for these requirements will become clear later. Note that (2.36) follows from (2.35).

Since $0 < h < 1$, it is easy to see that (2.35) holds for all $\delta > 0$ sufficiently small. For such small values of δ the last assumption is easy to achieve. It is easy to see that (2.36) takes place whenever (2.35) holds. Therefore, it remains to show that (2.34) takes place. Indeed, taking logarithms it follows that (2.34) is equivalent to

$$\ell(x) := x\log x + e^{2\delta}\cos\epsilon + 2\delta x - x < 0, \quad 0 < \delta < \epsilon, \quad x \in [e^{-3\delta}, 1]. \tag{2.37}$$

Now $\ell'(x) = \log x + 2\delta$. So $\ell'(e^{-2\delta}) = 0$ and $\ell''(x) = 1/x > 0, x > 0$. Consequently, ℓ is convex on $[0, 1]$ with a global minimum at $e^{-2\delta}$. To prove (2.37) it suffices to choose δ so that at the extreme points of $[e^{-3\delta}, 1]$ the the function is less than 0. Now, $\ell(1) = e^{\delta}\cos\epsilon + 2\delta - 1$. Since $\epsilon > 0$ is fixed, $\cos\epsilon < 1$ and $\ell(1) < 0$ for all sufficiently small $\delta > 0$. Analogously, $\ell(e^{-3\delta}) = -(\delta + 1)e^{-3\delta} + e^{2\delta}\cos\epsilon$, which is also < 0 for all sufficiently small δ. Choosing $\delta, 0 < \delta < \epsilon$ so that these two values are negative we know that at all intermediate points ℓ is also negative.

Finally, we choose $n_0 = n_0(\epsilon, \delta)$ so that for $n \ge n_0$ all finite poles of $r_{n,m}$ are in the set $\left\{z : \ |z| < e^{\delta/2}\right\} \setminus \Delta_\epsilon$ and the circle $|z| = e^{-\delta}$ contains no poles of $r_{n,m}$. Recall that we are considering the case when 1 is not a zero of q.

First we prove (2.32). Let $\rho = e^{-\delta} < 1$. According to (2.8)

$$B(F_n)(z) = \frac{1}{2\pi i}\int\limits_{\Gamma_\rho} F_n(t)e^{z/t}\frac{dt}{t} =$$

$$\sum_{k=n}^{\infty}\frac{1}{2\pi i}\int\limits_{\Gamma_\rho}\frac{A_k t^{k+1}q_{k,m-m^*}(t)}{(q_{k,m}q_{k+1,m})(t)}(q_{n,m}\omega)(t)\left[e^{z/t} - S_{k-1}\left(\frac{z}{t}\right)\right]\frac{dt}{t} +$$

$$\sum_{k=n}^{\infty}\frac{1}{2\pi i}\int\limits_{\Gamma_\rho}\frac{A_k t^{k+1}q_{k,m-m^*}(t)}{(q_{k,m}q_{k+1,m})(t)}(q_{n,m}\omega)(t)S_{k-1}\left(\frac{z}{t}\right)\frac{dt}{t} = \Sigma_1(z) + \Sigma_2(z).$$

Let the curve γ be the boundary of the set $\left\{z: \ |z| \le e^{-\delta}\right\} \cup \Delta_\epsilon$. By the choice of ϵ and n_0 we have, for $n \ge n_0$,

$$\Sigma_1(z) = \sum_{k=n}^{\infty}\frac{1}{2\pi i}\int\limits_{\gamma}\frac{A_k t^{k+1}q_{k,m-m^*}(t)}{(q_{k,m}q_{k+1,m})(t)}(q_{n,m}\omega)(t)\left[e^{z/t} - S_{k-1}\left(\frac{z}{t}\right)\right]\frac{dt}{t} =$$

$$\sum_{k=n}^{\infty}\frac{A_k}{k!}z^k\frac{1}{2\pi i}\int\limits_{\gamma}\frac{tq_{k,m-m^*}(t)}{(q_{k,m}q_{k+1,m})(t)}(q_{n,m}\omega)(t)\frac{e^{z/t} - S_{k-1}(z/t)}{(1/k!)(z/t)^k}\frac{dt}{t} \tag{2.38}$$

If $|z| \le ne^{\delta}$ and $|\arg z| \le \epsilon$, we have, for $t \in \gamma$, either $|z/t| \le ne^{2\delta}$ and $|\arg(z/t)| \ge \epsilon$, or $|z/t| \le ne^{-\delta}$. Therefore, by (2.14) and (2.18), we obtain

$$\left|\frac{e^{z/t} - S_{k-1}(z/t)}{(1/k!)(z/t)^k}\right| \le C_4$$

uniformly for $z \in K_n(\epsilon, \delta)$, $t \in \gamma$, and $k \ge n$, where $K_n(\epsilon, \delta) = \{z: \ |z| \le ne^{\delta}, |\arg z| \le \epsilon\}$. Consequently, the inequality

$$|\Sigma_1(z)| \le C_5\sum_{k=n}^{\infty}\frac{|A_k^*|}{k!}|z|^k, \qquad z \in K_n(\epsilon, \delta), \tag{2.39}$$

follows from (2.38).

Now, we take up $\Sigma_2(z)$. To abbreviate we write r_k in place of $r_{k,m}$.

$$\Sigma_2(z) = \sum_{k=n}^{\infty}\frac{1}{2\pi i}\int\limits_{\Gamma_\rho}\left[(r_{k+1} - \mathfrak{f}) + (\mathfrak{f} - r_k)\right](t)(q_{n,m}\omega)(t)S_{k-1}\left(\frac{z}{t}\right)\frac{dt}{t}. \tag{2.40}$$

Since

$$\sum_{k=n}^{N-1}\frac{1}{2\pi i}\int\limits_{\Gamma_\rho}\left[(r_{k+1} - \mathfrak{f}) + (\mathfrak{f} - r_k)\right](t)(q_{n,m}\omega)(t)S_{k-1}\left(\frac{z}{t}\right)\frac{dt}{t} =$$

$$\frac{1}{2\pi i}\int\limits_{\Gamma_\rho}(q_{n,m}\mathfrak{f}\omega - p_{n,m}\omega)(t)S_{n-1}\left(\frac{z}{t}\right)\frac{dt}{t}$$

$$-\frac{1}{2\pi i}\int\limits_{\Gamma_\rho}(\mathfrak{f}-r_N)(t)q_{n,m}(t)S_{N-2}\left(\frac{z}{t}\right)\frac{dt}{t}$$

$$+\sum_{k=n}^{N-1}\frac{1}{2\pi i}\int\limits_{\Gamma_\rho}(\mathfrak{f}-r_{k+1})(t)(q_{n,m}\omega)(t)\left[S_k\left(\frac{z}{t}\right)-S_{k-1}\left(\frac{z}{t}\right)\right]\frac{dt}{t}$$

and since

$$(\mathfrak{f}-r_N)(t)\rightrightarrows 0, \qquad S_{N-2}(z/t)\rightrightarrows e^{z/t} \qquad \text{as } N\longrightarrow\infty,$$

uniformly for $t\in\Gamma_\rho$ and $|z|\le R$ (where $R\in(0,\infty)$ is any real number), we obtain from (2.40), taking account of the holomorphy of the function (2.28), the equality

$$\Sigma_2(z)=\sum_{k=n}^{\infty}\frac{1}{2\pi i}\int\limits_{\Gamma_\rho}(\mathfrak{f}-r_{k+1})(t)(q_{n,m}\omega)(t)\frac{1}{k!}\left(\frac{z}{t}\right)^k\frac{dt}{t}.$$

Hence, using Lemma 2.2.1 we have

$$|\Sigma_2(z)|\le C_6\sum_{k=n}^{\infty}\frac{|A_k^*|}{k!}|z|^k, \qquad z\in K_n(\epsilon,\delta) \tag{2.41}$$

Then, (2.32) follows from (2.39) and (2.41), for $z\in K_n(\epsilon,\delta)$.

Now we prove (2.33). Let $R=e^\delta>1$ and $\rho=e^{-\delta}<1$. By the choice of n_0, all the finite poles of r_n, $n\ge n_0$, lie on a compact subset surrounded by Γ_R. Using this fact and the equality

$$(p_{n,m}\omega)(z)=(r_{n_0}q_{n,m}\omega)(z)+\sum_{k=n_0}^{n-1}\frac{A_k z^{k+1}q^*_{k,m-m^*}(z)}{(q_{k,m}q_{k+1,m})(z)}(q_{n,m}\omega)(z),$$

we obtain

$$B(F_n)(z)=B(q_{n,m}\mathfrak{f}\omega)(z)-B(p_{n,m}\omega)(z)=$$

$$B(q_{n,m}\mathfrak{f}\omega)(z)-\frac{1}{2\pi i}\int\limits_{\Gamma_R}(r_{n_0}q_{n,m}\omega)(t)S_{n_0+m}\left(\frac{z}{t}\right)\frac{dt}{t}$$

$$-\sum_{k=n_0}^{n-1}\frac{1}{2\pi i}\int\limits_{\Gamma_R}\frac{A_k t^{k+1}q^*_{k,m-m^*}(z)}{(q_{k,m}q_{k+1,m})(z)}(q_{n,m}\omega)(z)S_{k+m}\left(\frac{z}{t}\right)\frac{dt}{t}.$$

In deducing these equalities we make use of the fact that integrals containing factors of the form $e^{z/t} - S_{k+m}(z/t)$ vanish since they have a zero at $t = \infty$ of order $k + m + 1$ which multiplied by the other factor under the integral sign, which is rational, produces a function holomorphic in the complement of Γ_R with a zero of order at least two at infinity. Then

$$B(F_n)(z) = B(q_{n,m}\mathfrak{f}\omega)(z) - I(z) - \Sigma(z). \tag{2.42}$$

Let $n > n_0' \geq e^{3\delta}(n_0 + m)$; then for $|z| \geq ne^{-\delta}$ and $t \in \Gamma_R$, we have the inequalities $|z/t| \geq ne^{-2\delta} \geq e^{\delta}(n_0 + m)$. Consequently,

$$\left|S_{n_0+m}\left(\frac{z}{t}\right)\right| \leq C_7 \frac{1}{(n_0 + m)!} \left|\frac{z}{t}\right|^{n_0+m}. \tag{2.43}$$

Therefore

$$|I(z)| \leq C_8 \frac{A^*_{n_0+m}}{(n_0 + m)!} |z|^{n_0+m} \qquad \text{for} \qquad |z| \geq ne^{-\delta}. \tag{2.44}$$

Now we estimate $\Sigma(z)$. Let

$$L = \left\{t : \ |t| = e^{\delta}, |\arg(t)| \geq 2\epsilon\right\} \cup \left\{t : |t| = e^{-\delta}, |\arg(t)| \geq 2\epsilon\right\}$$
$$\cup \left\{t : e^{-\delta} \leq |t| \leq e^{\delta}, |\arg(t)| = 2\epsilon\right\}.$$

Then

$$\Sigma(z) = \sum_{k=n_0}^{n-1} \frac{1}{2\pi i} \int\limits_{\Gamma_\rho} \frac{A_k t^{k+1} q^*_{k,m-m^*}}{(q_{k,m} q_{k+1,m})(t)} (q_{n,m}\omega)(t) S_{k+m}\left(\frac{z}{t}\right) \frac{dt}{t}$$

$$+ \sum_{k=n_0}^{n-1} \frac{1}{2\pi i} \int\limits_{L} \frac{A_k t^{k+1} q^*_{k,m-m^*}}{(q_{k,m} q_{k+1,m})(t)} (q_{n,m}\omega)(t) S_{k+m}\left(\frac{z}{t}\right) \frac{dt}{t}$$

$$= \Sigma_1(z) + \Sigma_2(z).$$

Moreover, since the function (2.28) is holomorphic, we have

$$\Sigma_1(z) = \sum_{k=n_0}^{n-1} \frac{1}{2\pi i} \int\limits_{\Gamma_\rho} \left[(r_{k+1} - \mathfrak{f}) + (\mathfrak{f} - r_k)\right](t)(q_{n,m}\omega)(t) S_{k+m}\left(\frac{z}{t}\right) \frac{dt}{t}$$

$$= \frac{1}{2\pi i} \int\limits_{\Gamma_\rho} (\mathfrak{f} - r_{n_0})(t)(q_{n,m}\omega)(t) S_{n_0+m}\left(\frac{z}{t}\right) \frac{dt}{t}$$

$$+\sum_{k=n_0}^{n-2} \frac{1}{2\pi i} \int\limits_{\Gamma_\rho} (\mathfrak{f} - r_{k+1})(t)(q_{n,m}\omega)(t) \frac{1}{(k+m)!} \left(\frac{z}{t}\right)^{k+m} \frac{dt}{t}. \tag{2.45}$$

Let $n \geq n_0'$. Then, by (2.43), Lemma 2.2.1, and (2.16), we obtain

$$|\Sigma_1(z)| \leq C_9 \sum_{k=n_0}^{n-1} \frac{A_k^*}{k!} |z|^k, \qquad ne^{-\delta} \leq |z| \leq ne^{\delta}, \tag{2.46}$$

from (2.45).

Now we estimate $\Sigma_2(z)$. Let $\lambda = e^{-3\delta} < 1$. We rewrite $\Sigma_2(z)$ in the following form ($[x]$ denotes the integer part of x):

$$\Sigma_2(z) = \sum_{k=n_0}^{[\lambda n]-m} \frac{1}{2\pi i} \int\limits_L \frac{A_k t^{k+1} q^*_{k,m-m^*}}{(q_{k,m}q_{k+1,m})(t)} (q_{n,m}\omega)(t) S_{k+m}\left(\frac{z}{t}\right) \frac{dt}{t}$$

$$- \sum_{k=[\lambda n]-m+1}^{n-1} \frac{1}{2\pi i} \int\limits_L \frac{A_k t^{k+1} q^*_{k,m-m^*}}{(q_{k,m}q_{k+1,m})(t)} (q_{n,m}\omega)(t) \left[e^{z/t} - S_{k+m}\left(\frac{z}{t}\right)\right] \frac{dt}{t}$$

$$+ \sum_{k=[\lambda n]-m+1}^{n-1} \frac{1}{2\pi i} \int\limits_L \frac{A_k t^{k+1} q^*_{k,m-m^*}}{(q_{k,m}q_{k+1,m})(t)} (q_{n,m}\omega)(t) e^{z/t} \frac{dt}{t}$$

$$= \Sigma_{2,1}(z) - \Sigma_{2,2}(z) + \Sigma_{2,3}(z).$$

Let $ne^{-\delta} \leq |z| \leq ne^{\delta}$. For $t \in L$ we have $|t| \leq e^{\delta}$; therefore $|z/t| \geq ne^{-2\delta}$. If $n_0 \leq k \leq [\lambda n] - m$, then

$$|z/t| \geq ne^{-2\delta} = \lambda n e^{\delta} \geq (k+m)e\delta.$$

Consequently, for the specified values of z, t, and k, we have by (2.15)

$$|S_{k+m}(z/t)| \leq \frac{C_{10}}{(k+m)!} \left|\frac{z}{t}\right|^{k+m}$$

and

$$|\Sigma_{2,1}(z)| < C_{11} \sum_{k=n_0}^{[\lambda n]-m} \frac{|A_k|}{k!} |z|^k, \qquad ne^{-\delta} \leq |z| \leq ne^{\delta}. \tag{2.47}$$

Now let $[\lambda n] - m + 1 \leq k \leq n - 1$; then for $|z| \leq ne^{\delta}$, $|\arg z| \leq \epsilon$ and $t \in L$, we have the inequalities $|z/t| \leq ne^{2\delta} < (k+m)e^{5\delta}$, and $|\arg(z/t)| \geq \epsilon$. Hence, by (2.18),

$$\left|e^{z/t} - S_{k+m}\left(\frac{z}{t}\right)\right| \leq \frac{C_{12}}{(k+m)!} \left|\frac{z}{t}\right|^{k+m},$$

and therefore by (2.16)

$$|\Sigma_{2,2}(z)| \leq C_{13} \sum_{k=[\lambda n]-m+1}^{n-1} \frac{A_k^*}{k!}|z|^k, \qquad ne^{-\delta} \leq |z| \leq ne^{\delta}, \quad |\arg z| \leq \epsilon. \tag{2.48}$$

For $t \in L$ and $|\arg z| \leq \epsilon$ we have $Re(z/t) \leq |z/t|\cos\epsilon \leq |z|e^{\delta}\cos\epsilon$. Consequently

$$|e^{z/t}| \leq e^{|z|e^{\delta}\cos\epsilon}.$$

Therefore

$$\overline{\lim}_{n\longrightarrow\infty}|e^{z/t}|^{1/n} \leq e^{e^{2\delta}\cos\epsilon}. \tag{2.49}$$

uniformly for $z \in K_n(\epsilon,\delta)$ and $t \in L$. Moreover,

$$\left|\frac{k!}{(z/t)^k}\right| \leq C\frac{k^{k+1}e^{-k}}{n^k e^{-2k\delta}} = Ck\left(\frac{k}{n}\right)^k e^{-k+2k\delta}$$

for $[\lambda n]-m+1 \leq k \leq n-1$, $|z| \geq ne^{-\delta}$ and $t \in L$. Therefore

$$\overline{\lim}_{n\longrightarrow\infty} \max\left|\frac{k!}{(z/t)^k}\right|^{1/n} \leq \max_{e^{-3\delta}\leq x\leq 1} x^x e^{-x+2x\delta} = x_0^{x_0}e^{-x_0+2x_0\delta} \tag{2.50}$$

It follows from (2.49), (2.50) and (2.34) that, uniformly for $z \in E_n(\epsilon,\delta) = \left\{z:\ |\arg(z)| \leq \epsilon,\ ne^{-\delta} \leq |z| \leq ne^{\delta}\right\}$, $t \in L$ and $[\lambda n]-m+1 \leq k \leq n-1$, we have

$$\overline{\lim}_{n\longrightarrow\infty}\left|\frac{e^{z/t}}{(1/k!)(z/t)^k}\right|^{1/n} \leq x_0^{x_0}e^{e^{2\delta}\cos\epsilon - x_0+2x_0\delta} < 1.$$

Consequently

$$|\Sigma_{2,3}(z)| \leq C_{14} \sum_{k=[\lambda n]-m+1}^{n-1} \frac{|A_k|}{k!}|z|^k, \qquad z \in E_n(\epsilon,\delta). \tag{2.51}$$

From (2.46), (2.47), (2.48) and (2.51) we obtain

$$|\Sigma(z)| \leq C_{15} \sum_{k=n_0}^{n-1} \frac{A_k^*}{k!}|z|^k, \qquad z \in E_n(\epsilon,\delta). \tag{2.52}$$

Next, if we take account of (2.27) and (2.35), we have, uniformly for $z \in E_n(\epsilon,\delta)$,

$$\overline{\lim}_{n\longrightarrow\infty}\left|\frac{B(q_{n,m}\mathfrak{f}\omega)(z)}{(A_{n-1}^*/(n-1)!)z^{n-1}}\right|^{1/n} \leq e^{he^{\delta}+\delta-1} < 1,$$

and therefore

$$|B(q_{n,m}\mathfrak{f}\omega)(z)| \leq C_(16) \left(A^*_{n-1}/(n-1)!\right) |z|^{n-1}, \qquad z \in E_n(\epsilon,\delta). \qquad (2.53)$$

Then (2.33) follows from (2.42), (2.44), (2.52) and (2.53).

Now we show that (2.31) follows from (2.32) and (2.33). Put

$$R_n = \frac{A^*_{n-1}}{(n-1)!}\frac{n!}{A^*_n} = n\frac{A^*_{n-1}}{A^*_n} = ne^{\theta_n},$$

where $\theta_n \to 0$ as $n \to \infty$. Then we obtain, from (2.32) and (2.33), respectively,

$$|B(F_n)(z)| \leq C_2 \sum_{k=n}^{\infty} \frac{A^*_k}{k!}|z|^k, \qquad ne^{-\delta} \leq |z| \leq R_n, \qquad |\arg(z)| \leq \epsilon,$$

and

$$|B(F_n)(z)| \leq C_3 \sum_{k=n_0}^{n-1} \frac{A^*_k}{k!}|z|^k, \qquad R_n \leq |z| \leq ne^{\delta}, \qquad |\arg(z)| \leq \epsilon.$$

Hence by the concavity of the sequence $\log(A^*_n/n!)$ it follows that

$$\left|\frac{B(F_n)(z)}{(A^*_n/n!)z^n}\right| \leq \frac{C_{17}}{1-|z/R_n|}, \qquad ne^{-\delta} \leq |z| \leq R_n, \qquad |\arg(z)| \leq \epsilon, \quad (2.54)$$

and

$$\left|\frac{B(F_n)(z)}{(A^*_n/n!)z^n}\right| \leq \frac{C_{18}}{|z/R_n|-1}, \qquad R_n \leq |z| \leq ne^{\delta}, \qquad |\arg(z)| \leq \epsilon. \quad (2.55)$$

From (2.54) and (2.55) we obtain

$$\left|\frac{B(F_n)(z)}{(A^*_n/n!)z^n}\right| \leq C_{19}, \qquad ne^{-\delta} \leq |z| \leq ne^{\delta}, \qquad |\arg(z)| \leq \frac{\epsilon}{2},$$

from which (2.31) follows. This completes the proof of Proposition (2.2.4). ■

Now we proceed directly to the proof of Lemma 2.2.2.

Proof. It follows from (2.29) and (2.31) that

$$\max_{|z|\leq ne^{\delta}} \max_{|\arg(z)|\leq\delta} \left|\frac{B(F_n)(z)}{(A^*_n/n!)z^n}\right| = \mathcal{O}(1), \qquad n \longrightarrow \infty \qquad (2.56)$$

We write (2.27) in the form

$$F_n(re^{i\varphi}) = \frac{1}{r}\int\limits_0^{ne^\delta} B(F_n)(\rho e^{i\varphi})e^{-\rho/r}d\rho + \frac{1}{r}\int\limits_{ne^\delta}^{\infty} B(q_{n,m}\mathfrak{f}\omega)(\rho e^{i\varphi})e^{-\rho/r}d\rho -$$

$$\frac{1}{r}\int\limits_{ne^\delta}^{\infty} B(p_{n,m}\omega)(\rho e^{i\varphi})e^{-\rho/r}d\rho =$$

$$I_{n,1}(r,\varphi) + I_{n,2}(r,\varphi) - I_{n,3}(r,\varphi),$$

where $e^{-\delta} \leq e^\delta$ and $|\varphi| \leq \delta$. From (2.56) we obtain

$$|I_{n,1}(r,\varphi)| \leq C_{20}\frac{A_n^*}{n!}\int\limits_0^{ne^\delta} \rho^n e^{-\rho/r}\frac{d\rho}{r} \leq C_{20}\frac{A_n^*}{n!}r^n\int\limits_0^{\infty} x^n e^{-x}dx = C_{20}A_n^* r^n. \tag{2.57}$$

Moreover, by (2.25) and (2.36),

$$|I_{n,2}(r,\varphi)| \leq C_{21}\int\limits_{ne^\delta}^{\infty} e^{(h-1/r)\rho}d\rho \leq C_{21}\int\limits_{ne^\delta}^{\infty} e^{(h-e^{-\delta})\rho}d\rho = C_{22}e^{(h-e^{-\delta})ne^\delta},$$

and consequently, taking account of (2.35), we have

$$\overline{\lim}_{n\longrightarrow\infty} |I_{n,2}(r,\varphi)|^{1/n} \leq e^{(h-e^{-\delta})e^\delta} = e^{he^\delta - 1} < e^{-\delta}.$$

Since

$$\underline{\lim}_{n\longrightarrow\infty}(A_n^* r^n)^{1/n} \geq e^{-\delta}$$

for $e^{-\delta} \leq r \leq e^\delta$, we have

$$|I_{n,2}(r,\varphi)| \leq C_{23}A_n^* r^n, \qquad e^{-\delta \leq r \leq e^\delta},\ |\epsilon| \leq \delta. \tag{2.58}$$

Now we estimate $I_{n,3}(r,\varphi)$. Using (2.30) and property (i) of A_n^*, we obtain

$$|I_{n,2}(r,\varphi)| \leq C_{24}\frac{A_{n+m}^*}{(n+m)!}\int\limits_{ne^\delta}^{\infty} \rho^{n+m}e^{(-\rho/r)}\frac{d\rho}{r} \leq$$

$$C_{24}\frac{A_{n+m}^*}{(n+m)!}r^{n+m}\int\limits_0^{\infty} x^{n+m}e^{-x}dx \leq C_{25}A_n^* r^n, \qquad e^{-\delta} \leq r \leq e^\delta. \tag{2.59}$$

It follows from (2.57)-(2.59) that

$$|F_n(re^{i\varphi})| \le C_(26)A_n^* r^n, \qquad e^{-\delta} \le r \le e^{\delta},\ |\varphi| \le \delta.$$

Thus Lemma 2.2.2 is proved under the condition $z_0 \neq a_j$, $j = 1, \cdots, m$.

In the general case, we obtain (2.11) for points near z_0. With Lemma 2.2.1 this yields the uniform boundedness of the family $F_n(z)/A_n^* z^n$ of holomorphic functions on the boundary of the set

$$K_\delta = \{z :\ e^{-\delta} \le |z| \le e^{\delta}, |\arg(z) - \arg(z_0)| \le \delta\}.$$

By the maximum principle for holomorphic functions we obtain the uniform boundedness of this family on the whole set K_δ, that is to say, in a neighborhood of z_0. Therefore we have proved Lemma 2.2.2. ■

§ 2.3. Inverse results.

In the sequel $\mathrm{dist}(\zeta, B_n)$ denotes the distance from a point ζ to a set B_n. Let $\mathcal{P}_{n,m}(\mathfrak{f}) = \{\zeta_{n,1}, \cdots, \zeta_{n,m_n}\}$ be the set of zeros of $q_{n,m}$ and the points are enumerated so that

$$|\zeta_{n,1} - \zeta| \le \cdots \le |\zeta_{n,m_n} - \zeta|.$$

We say that $\lambda = \lambda(\zeta)$ points of $\mathcal{P}_{n,m}$ tend to ζ if

$$\lim_{n\to\infty} |\zeta_{n,\lambda} - \zeta| = 0, \qquad \limsup_{n\to\infty} |\zeta_{n,\lambda+1} - \zeta| > 0.$$

By convention, $\limsup_{n\to\infty} |\zeta_{n,k} - \zeta| > 0$ for $k > \liminf_{n\to\infty} m_n$.

Theorem 2.3.1. *Let $\mathfrak{f}$ be a formal power series as in* (1.6)*. Fix $m \ge m^* \ge 1$. Assume that $0 < R_m^*(\mathfrak{f}) < +\infty$. Suppose that*

$$\lim_{n\to\infty} \mathrm{dist}(\zeta, \mathcal{P}_{n,m}(\mathfrak{f})) = 0.$$

Let $\mathcal{Z}_n(\mathfrak{f})$ be the set of zeros of $q^*_{n,m-m^*}$. If $|\zeta| > R_m^*(\mathfrak{f})$, then

$$\lim_{n\in\Lambda} \mathrm{dist}(\zeta, \mathcal{Z}_n(f)) = 0 \tag{2.60}$$

where Λ is any infinite sequence of indices verifying (iv) in the regularization of $(A_{n,m})_{n\ge m}$. If $|\zeta| < R_m^*(\mathfrak{f})$, then either (2.60) *takes place or ζ is a pole of $\mathfrak{f}$ of order greater or equal to $\lambda(\zeta)$. If*

$$\lim_{n\to\infty} q_{n,m} = q_m, \qquad \deg q_m = m, \qquad q_m(0) \neq 0 \qquad \text{and} \qquad |\zeta| = R_m^*(f),$$

then we have either (2.60) *or ζ is a singular point of $\mathfrak{f}$ and lies on the closure of $D_m^*(\mathfrak{f})$.*

Proof. Without loss of generality, we can assume that $R_m^*(\mathfrak{f}) = 1$. The general case reduces to it with the change of variables $z \to z/R_m^*(\mathfrak{f})$. Assume that $|\zeta| \neq 1$ and ζ is a regular point of $\mathfrak{f}$ should $|\zeta| < 1$. Choose $\delta > 0$ such that $|\zeta| > e^{\delta}$ or $|\zeta| < e^{-\delta}$ depending on whether $|\zeta| > 1$ or $|\zeta| < 1$, respectively. Let $q_{n,m}(\zeta_n) = 0$, $\lim_{n\to\infty} \zeta_n = \zeta$.

Evaluating at ζ_n, using (2.2), if $|\zeta| > 1$ or (2.3), when $|\zeta| < 1$, and taking (iv) of Theorem 2.1.1 into account, it follows that

$$\left|p_{n,m}(\zeta_n)/A_{n,m}^*\zeta_n^n\right| \leq C_1, \qquad n \geq 0, \qquad n \in \Lambda,$$

where C_1 is some constant and Λ is the sequence of indices which appears in the regularization of $(A_{n,m})_{n\geq m}$. (In the sequel $C_1, C_2, \cdots$ denote constants which do not depend on n.) However, from (1.22) it follows that

$$p_{n,m}(\zeta_n)/A_{n,m}^*\zeta_n^n = -\zeta_n^{1-\lambda_n-\lambda_{n+1}} q_{n,m-m^*}^*(\zeta_n)/q_{n+1,m}(\zeta_n),$$

which combined with the previous inequality gives

$$|q_{n,m-m^*}^*(\zeta_n)| \leq C_2|q_{n+1,m}(\zeta_n)|, \qquad n \geq n_0, \qquad n \in \Lambda.$$

Therefore, (2.60) takes place.

If $|\zeta| = 1$ and ζ is a regular point the proof of (2.60) is the same as for the case when $|\zeta| \neq 1$. In this case use (2.3) on a closed neighborhood of ζ contained in $G \supset D_m^*(\mathfrak{f}) \setminus \mathcal{P}(\mathfrak{f})$.

Now, assume that $|\zeta| < 1$ and $\limsup_{n\in\Lambda} \operatorname{dist}(\zeta, \mathcal{Z}_n(\mathfrak{f})) > 0$. Then, ζ is a singular point of $\mathfrak{f}$. Since $D_m^*(\mathfrak{f}) \subset D_m(\mathfrak{f})$ according to Theorem 1.3.1, ζ must be a pole of $\mathfrak{f}$. Let τ be the order of the pole of $\mathfrak{f}$ at ζ. Let $\omega(z) = (z-\zeta)^{\tau}$ and $F = \omega\mathfrak{f}$. Notice that $F(\zeta) \neq 0$. Using (2.3) and (iv), it follows that there exists a closed disk U_r centered at ζ of radius r sufficiently small so that

$$\max_{U_r}\left|\frac{(q_{n,m}F - p_{n,m}\omega)(z)}{A_{n,m}^* z^n}\right| \leq C_3, \qquad n \geq n_0, \qquad n \in \Lambda. \tag{2.61}$$

Suppose that $\tau < \lambda(\zeta)$. Since $\sigma_1 - \lim_{n\to\infty} r_{n,m} = \mathfrak{f}$ (see Theorem 1.3.1), it follows that for each $n \in \mathbb{Z}_+$ there exists a zero η_n of $p_{n,m}$ such that $\lim_{n\to\infty} \eta_n = \zeta$. Take $r > 0$ sufficiently small so that $\min_{U_r} |F(z)| > 0$. Substituting η_n in (2.61), we have

$$|q_{n,m}(\eta_n)/A_{n,m}^*\eta_n^n| \leq C_4, \qquad n \geq n_0, \qquad n \in \Lambda,$$

and taking into account that (1.22) leads to

$$|q_{n,m}(\eta_n)/A_{n,m}^*\eta_n^n| = -\eta_n^{1-\lambda_n-\lambda_{n+1}} q_{n,m-m^*}^*(\eta_n)/p_{n+1,m}(\eta_n),$$

we obtain

$$|q^*_{n,m-m^*}(\eta_n)| \leq C_5 |p_{n+1,m}(\eta_n)|, \qquad n \geq n_0, \qquad n \in \Lambda.$$

Since $\limsup_{n\in\Lambda} \operatorname{dist}(\zeta, \mathcal{Z}_n(\mathfrak{f})) > 0$, it follows that

$$\lim_{n\in\Lambda'} |p_{n+1,m}(\eta_n)| > 0, \tag{2.62}$$

for some subsequence $\Lambda' \subset \Lambda$.

The normalization (1.20) imposed on $q_{n,m}$ implies that for any compact $K \subset \mathbb{C}$ we have $\sup_n \max_K |q_{n,m}(z)| \leq C_6$. So, any sequence $(q_{n,m})_{n\in I}$, $I \subset \mathbb{Z}_+$, contains a subsequence which converges uniformly on any compact subset of $\mathbb{C}$. This, combined with $\sigma_1 - \lim_{n\to} r_{n,m} = \mathfrak{f}$ in $D^*_m(\mathfrak{f})$, and the assumption that $\tau < \lambda(\zeta)$ imply that there exists a sequence of indices $\Lambda'' \subset \Lambda'$ such that $\lim_{n\in\Lambda''} p_{n+1,m} = F_1$ uniformly on a closed neighborhood of ζ, where F_1 is analytic at ζ and $F_1(\zeta) = 0$ (see [16, Lemma 1] where it is shown that under adequate assumptions uniform convergence on compact subsets of a region can be derived from convergence in 1-dimensional Hausdorff content). This contradicts (2.62). Thus, $\tau \geq \lambda(\zeta)$ as claimed. ∎

Since $\deg q^*_{n,m-m^*} \leq m - m^*$ for all $n \geq m$. Should $\lim_{n\to\Lambda} q^*_{n,m-m^*} = q^*_m$ then $\deg q^*_m \leq m - m^*$. This places some restriction on the number of zeros of q_m which verify (2.60); that is, at most $m - m^*$ distinct zeros of q_m can fulfill (2.60). In particular we have

***Corollary* 2.3.1.** *Suppose that $\lim_n q_{n,m} = q_m, \deg q_m = m, q_m(0) \neq 0$, all the zeros of q_m are distinct and $R^*_m(\mathfrak{f}) < +\infty$. Then at least m^* of the zeros of q_m are singular points of $\mathfrak{f}$ and lie in the closure of $D^*_m(\mathfrak{f})$, those lying in $D^*_m(\mathfrak{f})$ are simple poles.*

Proof. By Theorem 1.3.2, we have $0 < R_0(\mathfrak{f}) \leq R^*_m(\mathfrak{f})$. We know that $\deg q^*_{n,m-m^*} \leq m - m^*$ for all $n \geq m$. In particular, this implies that for each $n \in \Lambda$ the set $\mathcal{Z}_n(\mathfrak{f})$ has at most $m - m^*$ points. We can assume that $R^*_m(\mathfrak{f}) = 1$. Suppose that less that m^* zeros of q_m are singular points of $\mathfrak{f}$ in the closure of $D^*_m(\mathfrak{f})$. This means that $m - m^* + 1$ of them are either regular points of $\mathfrak{f}$ in the closure of $D^*_m(\mathfrak{f})$ or have absolute value greater than $R^*_m(\mathfrak{f})$. According to Theorem 2.3.1 there exists a subsequence of indices $\Lambda' \subset \Lambda$ such that

$$\lim_{n\to\Lambda'} q^*_{n,m-m^*} = C q^*_m, \qquad \deg q^*_m \geq m - m^* + 1,$$

where C is a constant different from zero. This is clearly impossible. On the other hand, according to Theorem 1.3.2 those zeros lying in $D^*_m(\mathfrak{f})$ are simple poles as we claimed. ∎

Now, suppose we know that

$$\lim_{n\to\infty} q^*_{n,m-m^*} = q^*_m \tag{2.63}$$

and $\mathcal{Z}(\mathfrak{f})$ is the set of zeros of q^*_m. Let $\mathcal{P}(\mathfrak{f})$ denote the set of zeros of q_m.

***Corollary* 2.3.2.** *Suppose that* $\lim_n q_{n,m} = q_m, \deg q_m = m, q_m(0) \neq 0$ *and* (2.63) *take place. Then all the points in* $\mathcal{P}(\mathfrak{f}) \setminus \mathcal{Z}(\mathfrak{f})$ *are singular points of* $\mathfrak{f}$.

This corollary is a direct consequence of Theorem 2.3.1. Notice that when $m = m^*$ then $q^*_{n,m-m^*} \equiv 1$; consequently, $\mathcal{Z}(\mathfrak{f}) = \emptyset$ and the corollary reduces to Suetin's Theorem.

In order to improve this corollary it would be convenient to establish a closer connection between the zeros of q_m and the accumulation points of the zeros of $q^*_{n,m-m^*}$, at least under assumption (2.63). Numerical evidence obtained in Chapter 5 suggests that the following statements hold true.

Let ζ be a zero of q_m of multiplicity τ. Assume that either $|\zeta| > R^*_m$ or $|\zeta| \leq R^*_m$ and it is a regular point of $\mathfrak{f}$; then, ζ is a zero of q^*_m of multiplicity $\geq \tau$. Additionally, if $|\zeta| < R^*_m$ and it is a pole of $\mathfrak{f}$ of order τ^* then it must be a zero of q^*_m of multiplicity $\geq \tau - \tau^*$.

The validity of these statements would allow to weaken the assumption regarding the simplicity of the zeros of q_m in Corollary 2.3.1 and the results of Section 2.5.

§ 2.4. System poles are strong attractors.

Fix $(\mathbf{f}, \mathbf{m})$ and $\zeta \in \mathbb{C}^*$. Let $\zeta_{n,1}, \ldots, \zeta_{n,\ell_n}, 0 \leq \ell_n \leq |\mathbf{m}|$, be the zeros of $q_{n,\mathbf{m}}$ indexed in increasing distance from ζ. That is

$$|\zeta - \zeta_{n,1}| \leq |\zeta - \zeta_{n,2}| \leq \cdots \leq |\zeta - \zeta_{n,\ell_n}|.$$

Following A.A. Gonchar in [17], we define two characteristic values. Set $\lambda(\zeta) := \nu$ if

$$\lim_{n\to\infty} |\zeta - \zeta_{n,\nu}| = 0, \qquad \limsup_{n\to\infty} |\zeta - \zeta_{n,\nu+1}| > 0$$

(for $\nu > \ell_n$ by convention $|\zeta - \zeta_{n,\nu}| := 1$, and when $\limsup_{n\to\infty} |\zeta - \zeta_{n,1}| > 0$, we take $\lambda(\zeta) = 0$). Analogously, $\mu(a) := \nu$ if

$$\limsup_{n\to\infty} |\zeta - \zeta_{n,\nu}|^{1/n} < 1 \qquad \limsup_{n\to\infty} |\zeta - \zeta_{n,\nu+1}|^{1/n} \geq 1.$$

We start out proving the following direct type result.

***Theorem* 2.4.1.** *Let ζ be a system pole of $(\mathbf{f},\mathbf{m})$ of order τ then $\mu(\zeta)\geq\tau$.*

Proof. For each $n\geq|\mathbf{m}|$, let $Q_{n,\mathbf{m}}$ be the polynomial $q_{n,\mathbf{m}}$ normalized so that

$$\sum_{k=0}^{|\mathbf{m}|}|\lambda_{n,k}|=1,\qquad Q_{n,\mathbf{m}}(z)=\sum_{k=0}^{|\mathbf{m}|}\lambda_{n,k}z^k. \tag{2.64}$$

This normalization entails that for any fixed $j\in\mathbb{Z}_+$ the sequence of polynomials $(Q^{(j)}_{n,\mathbf{m}})_{n\geq|\mathbf{m}|}$ is uniformly bounded on each compact subset of $\mathbb{C}$.

Let ζ be a system pole of order τ of $(\mathbf{f},\mathbf{m})$. Consider a polynomial combination g_1 of type (1.17) that is analytic on a neighborhood of $\overline{D}_{|\zeta|}$ except for a simple pole $z=\zeta$ and verifies that $R_1(g_1)=R_{\zeta,1}(\mathbf{f},\mathbf{m})(=\mathbf{r}_{\zeta,\mathbf{1}}(\mathbf{f},\mathbf{m}))$. Then we have

$$g_1=\sum_{k=1}^{|\mathbf{m}|}p_{k,1}\mathfrak{f}^k,\qquad \deg p_{k,1}<m_k,\qquad k=1,\ldots,|\mathbf{m}|,$$

and

$$Q_{n,\mathbf{m}}(z)h_1(z)-(z-\zeta)\sum_{k=1}^{|\mathbf{m}|}p_{k,1}(z)P_{n,\mathbf{m},\mathbf{k}}(z)=Az^{n+1}+\cdots,$$

where $h_1(z)=(z-\zeta)g_1(z)$. Hence, the function

$$\frac{Q_{n,\mathbf{m}}(z)h_1(z)}{z^{n+1}}-\frac{z-\zeta}{z^{n+1}}\sum_{k=1}^{|\mathbf{m}|}p_{k,1}(z)P_{n,\mathbf{m},\mathbf{k}}(z)$$

is analytic on $D_1(g_1)$. Take $0<r<R_1(g_1)$, and set $\Gamma_r=\{z\in\mathbb{C}:\ |z|=r\}$. Using Cauchy's formula, we obtain

$$Q_{n,\mathbf{m}}(z)h_1(z)-(z-\zeta)\sum_{k=1}^{|\mathbf{m}|}p_{k,1}P_{n,\mathbf{m},\mathbf{k}}(z)=\frac{1}{2\pi i}\int_{\Gamma_r}\frac{z^{n+1}}{\omega^{n+1}}\frac{Q_{n,\mathbf{m}}(\omega)h_1(\omega)}{\omega-z}d\omega,$$

for all z with $|z|<r$, since $\deg\sum_{k=1}^{|\mathbf{m}|}p_{k,1}P_{n,\mathbf{m},\mathbf{k}}<n$. In particular, taking $z=\zeta$ in the previous formula, we obtain

$$Q_{n,\mathbf{m}}(\zeta)h_1(\zeta)=\frac{1}{2\pi i}\int_{\Gamma_r}\frac{\zeta^{n+1}}{\omega^{n+1}}\frac{Q_{n,\mathbf{m}}(\omega)h_1(\omega)}{\omega-\zeta}d\omega. \tag{2.65}$$

Then

$$\limsup_{n\to\infty}|Q_{n,\mathbf{m}}(\zeta)h_1(\zeta)|^{1/n}\leq\frac{|\zeta|}{r}.$$

Using that $h_1(\zeta) \neq 0$ and making r tend to $R_1(g_1)$, we have

$$\limsup_{n\to\infty} |Q_{n,\mathbf{m}}(\zeta)|^{1/n} \leq \frac{|\zeta|}{R_{\zeta,1}(\mathbf{f},\mathbf{m})} < 1.$$

Now, we use induction to prove that for each $s = 0, \ldots, \tau - 1$

$$\limsup_{n\to\infty} |Q^{(s)}_{n,\mathbf{m}}(\zeta)|^{1/n} \leq \frac{|\zeta|}{R_{\zeta,s+1}(\mathbf{f},\mathbf{m})} \leq \frac{|\zeta|}{R_{\zeta}(\mathbf{f},\mathbf{m})}. \tag{2.66}$$

For $s = 0$ the property is true as was shown above. Suppose that

$$\limsup_{n\to\infty} |Q^{(j)}_{n,\mathbf{m}}(\zeta)|^{1/n} \leq \frac{|\zeta|}{R_{\zeta,j+1}(\mathbf{f},\mathbf{m})}, \qquad j = 0, 1, \ldots, s-2, \tag{2.67}$$

where $R_{\zeta,j+1}(\mathbf{f},\mathbf{m}) = \min_{k=1,\ldots,j+1} r_{\zeta,k}(\mathbf{f},\mathbf{m})$. Let us prove that (2.67) holds for $j = s-1$, with $s \leq \tau$.

Consider a polynomial combination g_s of type (1.17) that is analytic on a neighborhood of $\overline{D}_{|\zeta|}$ except for a pole of order s at $z = \zeta$ and verifies that $R_s(g_s) = r_{\zeta,s}(\mathbf{f},\mathbf{m})$. Then,

$$g_s = \sum_{k=1}^{|\mathbf{m}|} p_{k,s}\mathfrak{f}^k, \qquad \deg p_{k,s} < m_k, \qquad k = 1, \ldots, |\mathbf{m}|.$$

Set $h_s(z) = (z-\zeta)^s g_s(z)$. Reasoning as in the previous case, the function

$$\frac{Q_{n,\mathbf{m}}(z)h_s(z)}{z^{n+1}(z-\zeta)^{s-1}} - \frac{z-\zeta}{z^{n+1}} \sum_{k=1}^{|\mathbf{m}|} p_{k,s}(z)P_{n,\mathbf{m},\mathbf{k}}(z)$$

is analytic on $D_s(g_s) \setminus \{\zeta\}$. Set $P_s = \sum_{k=1}^{|\mathbf{m}|} p_{k,s}P_{n,\mathbf{m},\mathbf{k}}$. Fix an arbitrary compact set $\mathcal{K} \subset (D_s(g_s) \setminus \{\zeta\})$. Take $\delta > 0$ sufficiently small and $0 < r < R_s(g_s)$ with $\mathcal{K} \subset D_r$. Using Cauchy's integral formula and the residue theorem, since $\deg P_s < n$ for all $z \in \mathcal{K}$ we have

$$\frac{Q_{n,\mathbf{m}}(z)h_s(z)}{(z-\zeta)^{s-1}} - (z-\zeta)P_s(z) = I_n(z) - J_n(z), \tag{2.68}$$

where

$$I_n(z) = \frac{1}{2\pi i}\int_{\Gamma_r} \frac{z^{n+1}}{\omega^{n+1}} \frac{Q_{n,\mathbf{m}}(\omega)h_s(\omega)}{(\omega-\zeta)^{s-1}(\omega-z)} d\omega$$

and

$$J_n(z) = \frac{1}{2\pi i}\int_{|\omega-\zeta|=\delta} \frac{z^{n+1}}{\omega^{n+1}} \frac{Q_{n,\mathbf{m}}(\omega)h_s(\omega)}{(\omega-\zeta)^{s-1}(\omega-z)} d\omega.$$

The first integral I_n is estimated as in (2.65) to obtain

$$\limsup_{n\to\infty} \|I_n(z)\|_{\mathcal{K}}^{1/n} \leq \frac{\|z\|_{\mathcal{K}}}{R_s(g_s)} = \frac{\|z\|_{\mathcal{K}}}{r_{\zeta,s}(\mathbf{f},\mathbf{m})}. \tag{2.69}$$

For $J_n(z)$, as $\deg Q_{n,\mathbf{m}} \leq |\mathbf{m}|$ write

$$Q_{n,\mathbf{m}}(\omega) = \sum_{j=0}^{|\mathbf{m}|} \frac{Q_{n,\mathbf{m}}^{(j)}(\zeta)}{j!}(\omega-\zeta)^j.$$

Then

$$J_n(z) = \sum_{j=0}^{s-2} \frac{1}{2\pi i} \int_{|\omega-\zeta|=\delta} \frac{z^{n+1}}{\omega^{n+1}} \frac{h_s(\omega)}{(\omega-\zeta)^{s-1-j}} \frac{Q_{n,\mathbf{m}}^{(j)}(\zeta)}{j!(\omega-z)} d\omega. \tag{2.70}$$

Using the induction hypothesis, (2.67), and making δ tend to zero, we obtain

$$\limsup_{n\to\infty} \|J_n(z)\|_{\mathcal{K}}^{1/n} \leq \frac{\|z\|_{\mathcal{K}}}{|\zeta|} \frac{|\zeta|}{R_{\zeta,s-1}(\mathbf{f},\mathbf{m})} = \frac{\|z\|_{\mathcal{K}}}{R_{\zeta,s-1}(\mathbf{f},\mathbf{m})},$$

which, together with (2.68) and (2.69), gives

$$\limsup_{n\to\infty} \|Q_{n,\mathbf{m}}(z)h_s(z) - (z-\zeta)^s P_s(z)\|_{\mathcal{K}}^{1/n} \leq \frac{\|z\|_{\mathcal{K}}}{R_{\zeta,s-1}(\mathbf{f},\mathbf{m})}. \tag{2.71}$$

As the function inside the norm in (2.71) is analytic in $D_s(g_s)$, inequality (2.71) also holds for any compact set $\mathcal{K} \subset D_s(g_s)$. Moreover, we can differentiate $s-1$ times that function and the inequality still holds true by virtue of Cauchy's integral formula. So, taking $z=\zeta$ in (2.71) for the differentiated version, we obtain

$$\limsup_{n\to\infty} |(Q_{n,\mathbf{m}}h_s)^{(s-1)}(\zeta)|^{1/n} \leq \frac{|\zeta|}{R_{\zeta,s}(\mathbf{f},\mathbf{m})}.$$

Using the Leibniz formula for higher derivatives of a product of two functions and the induction hypothesis (2.67), we arrive at

$$\limsup_{n\to\infty} |Q_{n,\mathbf{m}}^{(s-1)}(\zeta)|^{1/n} \leq \frac{|\zeta|}{R_{\zeta,s}(\mathbf{f},\mathbf{m})} \leq \frac{|\zeta|}{R_{\zeta}(\mathbf{f},\mathbf{m})}, \tag{2.72}$$

since $h_s(\zeta) \neq 0$. This complete the induction.

Now, let us prove the weaker statement that $\lambda(\zeta) \geq \tau$. It is sufficient to show that for any subsequence of indices Λ such that

$$\lim_{n\in\Lambda} Q_{n,\mathbf{m}} = Q_\Lambda,$$

Q_Λ is a non null polynomial with a zero at ζ of multiplicity $\geq \tau$. Indeed, $Q_\Lambda \not\equiv 0$ due to the normalization on the polynomials $Q_{n,\mathbf{m}}$. On the other hand,

$$Q_{n,\mathbf{m}}(z) = \sum_{k=0}^{|\mathbf{m}|} \frac{Q_{n,\mathbf{m}}^{(k)}(\zeta)}{k!}(z-\zeta)^k.$$

Using (2.67), and Weierstrass' theorem for the derivatives it follows that

$$\lim_{n\to\Lambda} Q_{n,\mathbf{m}}(z) = Q_\lambda(z) = \sum_{k=\tau}^{|\mathbf{m}|} \frac{Q_\Lambda^{(k)}(\zeta)}{k!}(z-\zeta)^k,$$

as needed, since $Q_\Lambda(\zeta) = Q_\Lambda^{(1)}(\zeta) = \cdots = Q_\Lambda^{(\tau-1)}(\zeta) = 0$.

Set $U_\varepsilon = \{z : |z-\zeta| < \varepsilon\}$. Let ε be sufficiently small so that $U_{2\varepsilon}$ contains no other system pole of $(\mathbf{f},\mathbf{m})$ except ζ. Let $\zeta_{n,1},\ldots,\zeta_{n,\lambda_n}$ be the zeros of $Q_{n,\mathbf{m}}$ contained in $U_{2\varepsilon}$. Since $\lambda(\zeta) \geq \tau$, we have $\tau \leq \lambda_n \leq |\mathbf{m}|$ for all sufficiently large n. In the sequel we only consider such values of n. Set

$$\widetilde{Q}_n(z) = \prod_{k=1}^{\lambda_n}(z-\zeta_{n,k}).$$

It is easy to see that the functions $\widetilde{Q}_n/Q_{n,\mathbf{m}}$ are holomorphic in $U_{2\varepsilon}$ and uniformly bounded on any compact subset of $U_{2\varepsilon}$; in particular on $\overline{U}_\varepsilon$. Therefore, for any $k \geq 0$ the sequence $\left(\widetilde{Q}_n/Q_{n,\mathbf{m}}\right)^{(k)}$ is uniformly bounded on $\overline{U}_\varepsilon$. Since

$$\widetilde{Q}_n = Q_{n,\mathbf{m}}\frac{\widetilde{Q}_n}{Q_{n,\mathbf{m}}},$$

from (4.6) it readily follows that for each $s = 0,\ldots,\tau-1$

$$\limsup_{n\to\infty} |\widetilde{Q}_n^{(s)}(\zeta)|^{1/n} \leq \frac{|\zeta|}{R_\zeta(\mathbf{f},\mathbf{m})} < 1. \tag{2.73}$$

Now, using (2.73) for $s = 0$ and the ordering imposed on the indexing of the zeros of $q_{n,\mathbf{m}}$ it follows that

$$\limsup_{n\to\infty} |\zeta - \zeta_{n,1}|^{1/n} < 1$$

so that $\mu(\zeta) \geq 1$. Let us assume that for each $j = 1,\ldots,k$ where $k \leq \tau - 1$,

$$\limsup_{n\to\infty} |\zeta - \zeta_{n,j}|^{1/n} < 1, \tag{2.74}$$

and let us show that it is also true for $k+1$. Consider $\widetilde{Q}_n^{(k)}(\zeta)$. One of the terms thus obtained is $\prod_{j=k+1}^{\lambda_n}(\zeta - \zeta_{n,j})$, each one of the other terms contains at least one factor of the $(\zeta - \zeta_{n,j}), j = 1, \ldots, k$. Combining (2.73) and (2.74) it follows that

$$\limsup_{n\to\infty} |\prod_{j=k+1}^{\lambda_n}(\zeta - \zeta_{n,j})|^{1/n} < 1,$$

and due to the ordering of indexes, we get

$$\limsup_{n\to\infty} |\zeta - \zeta_{n,k+1}|^{1/n} < 1.$$

Consequently, $\mu(\zeta) \geq \tau$ as we wanted to prove. ∎

§ 2.5. Applications to Hermite-Padé approximation.

Let $\mathbf{f} = (\mathfrak{f}^1, \mathfrak{f}^2, \ldots, \mathfrak{f}^d)$ and $\mathbf{m} = (m_1, \ldots, m_d)$ be given. Consider the sequence $(\mathbf{R}_{n,\mathbf{m}})$, $n \geq \max\{m_1, \ldots, m_d\}$, of Hermite-Padé approximants. In the rest of this section we assume that the sequence of commom denominators $(q_{n,\mathbf{m}})$, $n \geq \max\{m_1, \ldots, m_d\}$ verifies (1.19).

***Theorem* 2.5.1.** *Let* $\mathbf{f} = (\mathfrak{f}^1, \mathfrak{f}^2, \ldots, \mathfrak{f}^d)$ *and* $\mathbf{m} = (m_1, \ldots, m_d)$ *be given. Assume that* (1.19) *takes place and all the zeros of* $q_{\mathbf{m}}$ *are simple. Fix an integer* m^*, $1 \leq m^* \leq \max\{m_1, \ldots, m_d\}$. *Assume that for all* n *sufficiently large* $q_{n,\mathbf{m}}$ *is unique and* $\deg(q_{n,\mathbf{m}}) = |\mathbf{m}|$. *Let*

$$F = \sum_{k=1}^{d} p_k \mathfrak{f}^k, \qquad \deg p_k \leq m_k - m^*, \tag{2.75}$$

where the p_k *denote arbitrary fixed polynomials (by convention* $\deg p_k < 0$ *means that* $p_k \equiv 0$*). Then, the closure of* $D_{m^*-1}(F)$ *contains at least* m^* *singular points of* F *which are zeros of* $q_{\mathbf{m}}$ *and those lying in* $D_{m^*-1}(F)$ *are simple poles of* F. *In particular, all such zeros are system singularities of* $\mathbf{f}$.

Proof. In the first part of the proof it is not used that the zeros of $q_{\mathbf{m}}$ are simple. Multiplying each relation $a.2)$ in Definition 1.2.1 by p_k for $k = 1, \cdots, d$ and adding them up it follows that

$$q_{n,\mathbf{m}}(z)F(z) - P_{n,\mathbf{m}}(z) = A_{n,\mathbf{m}} z^{n+1} + \cdots \tag{2.76}$$

where $P_{n,\mathbf{m}}(z) = \sum_{k=1}^{d} p_k(z) p_{n,\mathbf{m}}(z)$ is of degree $n - m^*$. It follows that $P_{n,\mathbf{m}}/q_{n,\mathbf{m}}$ is an incomplete Padé approximation of type $(n, |\mathbf{m}|, m^*)$ with respect to F. From Theorem 1.3.2 it follows that $0 < R_0(F) < \infty$ and due to Theorem 1.3.1

$$\sigma_1 - \lim_{n\to\infty} \frac{P_{n,\mathbf{m}}}{q_{n,\mathbf{m}}} = F$$

on compact subsets of $D^*_{|\mathbf{m}|}(F)$, where $D^*_{|\mathbf{m}|}(F)$ is the disk of radius $R^*_{|\mathbf{m}|}(F)$ given by (1.23) relative to the function F and the indices $|\mathbf{m}|, m^*$.

In $D^*_{|\mathbf{m}|}(F)$, F contains only poles and according to [16, Lemma 1], each pole of F in $D^*_{|\mathbf{m}|}(F) \supset D_{m^*}(F)$ must be a zero of $q_{\mathbf{m}}$ (counting multiplicities). If $R^*_{|\mathbf{m}|}(F) > R_{m^*}(F)$ from the definition of $D_{m^*}(F)$ the closure of this region has at least m^* poles. There are two possibilities, either $D_{m^*}(F)$ has exactly m^* poles and whence the closure of $D_{m^*-1}(F)$ has exactly m^* poles or $D_{m^*-1}(F) = D_{m^*}(F)$ and their closure coincide from which it follows that the closure of $D_{m^*-1}(F)$ has at least m^* poles. So in this case the assertion of the theorem is true. Therefore, in the following we can assume that $R^*_{|\mathbf{m}|}(F) = R_{m^*}(F)$. As above, should $D_{m^*}(F)$ contain m^* poles, they all lie in the closure of $D_{m^*-1}(F)$, and the proof is complete.

Now, assume that $R^*_{|\mathbf{m}|}(F) = R_{m^*}(F)$ and $D_{m^*}(F)$ contains less than m^* poles of F; then, $R^*_{|\mathbf{m}|}(F) = R_{m^*}(F) = R_{m^*-1}(F)$. Let w be the polynomial of degree $\leq m^* - 1$ whose zeros are the poles of F in $D_{m^*-1}(F)$ (counting multiplicities). Multiplying (2.76) by w, we obtain

$$q_{n,\mathbf{m}}(z)(\omega F)(z) - \omega(z) P_{n,\mathbf{m}}(z) = A_{n,\mathbf{m}} z^{n+1} + \cdots ,$$

where $\deg(\omega P_{n,\mathbf{m}}) \leq n - 1$. Notice that $D_0(wF) = D_{m^*-1}(F)$ and that $wP_{n,\mathbf{m}}/q_{n,\mathbf{m}}$ is an incomplete Padé approximation of type $(n, |\mathbf{m}|, 1)$ of wF. From hypothesis, for all sufficiently large n, $q_{n,\mathbf{m}}$ is unique and $\deg(q_{n,\mathbf{m}}) = |\mathbf{m}|$, using [10, Lemma 3.2] we obtain that wF is not a polynomial. Then, using [10, Lemma 2.5] we conclude that $R_0(wF) < \infty$. Consequently, $R^*_{|\mathbf{m}|}(F) = R_{m^*}(F) = R_{m^*-1}(F) = R_0(wF) < 0$. Without loss generality, we can assume that $R^*_{|\mathbf{m}|}(F) = 1$. In the rest of the proof we use that the zeros of $q_{n,|\mathbf{m}|}$ are simple. Suppose that between the zeros of $q_{\mathbf{m}}$ lying in the closure of $D_{m^*-1}(F)$ less than m^* of them are singular points of F. Using Theorem 2.3.1 and that the sequence of polynomials $(q_{n,|\mathbf{m}|-m^*})_{n\geq|\mathbf{m}|}$ corresponding to the function F is uniformly bounded on compact sets, we deduce that there exists a sequence of indices $\Lambda' \subset \Lambda$, a constant $0 < C < \infty$, and a polynomial Q, $\deg(Q) > |\mathbf{m}| - m^*$, such that

$$\lim_{n\in\Lambda'} q^*_{n,|\mathbf{m}|-m^*} = CQ.$$

This is so because each zero of $q_{\mathbf{m}}$ in the closure of $D_{m^*-1}(F) = D^*_{|\mathbf{m}|}(F)$ which is a regular point of F and each zero lying outside the closure of

$D^*_{|\mathbf{m}|}(F)$ is a limit point of the zeros of $q_{n,|\mathbf{m}|-m^*}$, $n \in \Lambda$. This is clearly impossible because $\deg(q_{n,|\mathbf{m}|-m^*}) \leq |\mathbf{m}| - m^*$ for all n. Thus, F has at least m^* singularities in the closure of $D_{m^*-1}(F)$ as claimed. That they are system singularities of $\mathbf{f}$ follows from Definition 1.2.3. The proof is complete. ■

In the next chapter we use a different approach to study the system singularities of $\mathbf{f}$. In particular Theorem 3.2.1 supplements Theorem 2.5.1 giving a wide class of systems $\mathbf{f}$ for which all the zeros of $q_{\mathbf{m}}$ are system singularities of $\mathbf{f}$.

CHAPTER 3

Higher order recurrences and row sequences of Hermite-Padé approximation

§ 3.1. Background.

Let

$$\mathcal{P}_n = \{\zeta_{n,1}, \ldots, \zeta_{n,m}\}, \quad n \geq m,$$

denote the collection of zeros of α_n repeated according to their multiplicity where α_n is the polynomial (1.7) associated with the recurrence (1.1). Set

$$S = \sup_{N \geq m} \inf_{n \geq N} \{|\zeta| : \zeta \in \mathcal{P}_n\}$$

and

$$G = \inf_{N \geq m} \sup_{n \geq N} \{|\zeta| : \zeta \in \mathcal{P}_n\}.$$

The following result is a consequence of [10, Theorem 2.2]

***Theorem* 3.1.1.** *Assume that $S > 0$ and $G < +\infty$. Then, any nontrivial solution $\mathfrak{f}$ of (1.1) verifies $0 < c \leq R_0(\mathfrak{f}) \leq C < +\infty$, where c and C only depend on S and G.*

Proof. For the benefit of the reader, we establish a connection between the notation employed here and the paper [10, Theorem 2.2].

Let $\mathfrak{f}$ be any non-trivial solution of (1.1) and $t_n = t_n(\mathfrak{f})$ the polynomial part of degree $\leq n-1$ of $\alpha_n \mathfrak{f}$. Due to (1.1) and (1.8) it follows that

$$(\alpha_n \mathfrak{f} - t_n)(z) = \mathcal{O}\left(z^{n+1}\right).$$

This means that the rational function t_n/α_n is what in [10] is called an (n, m, m^*) incomplete Padé approximation of $\mathfrak{f}$ with $m^* = 1$.

In [10], α_n is denoted $q_{n,m}$, t_n is denoted $p_{n,m}$, λ_n is the degree of the common zero which $p_{n,m}$ and $q_{n,m}$ may have at $z = 0$, $m_n = \deg(q_{n,m})$, and $\tau_n = \min\{n - m^* - \lambda_n - \deg(p_{n,m}), m - \lambda_n - m_n\}$. In our case, since $\alpha_n(0) = 1$ and $\deg(q_{n,m}) = m$, we have that $\lambda_n = 0, m_n = m, \tau_n = 0$, and $m^* = 1$. Thus, the assumptions in (i) and (ii) of [10, Theorem 1.2] are fulfilled and, therefore, $0 < R_0(\mathfrak{f}) < \infty$ as claimed. The proof gives lower and upper estimates of $R_0(\mathfrak{f})$ depending on S and G which imply the last assertion of Theorem 3.1.1. ■

This means that under such general conditions, every solution of (1.1) has a singularity in the annulus $\{z : c \leq |z| \leq C\}$. A more precise result may be given when the sequence of polynomials $(\alpha_n)_{n\geq m}$, has a limit.

In the following, we assume that (1.2) takes place and

$$\lim_{n\to\infty} \alpha_n(z) := \alpha(z) = \prod_{k=1}^{m} \left(1 - \frac{z}{\zeta_k}\right), \qquad \deg(\alpha) = m. \tag{3.1}$$

According to (1.3), we have $\alpha(z) = z^m p(1/z)$; therefore, the zeros of α and of the characteristic polynomial p are reciprocals one of the other.

Putting together Theorems 1 and 2 of V. I Buslaev in [3], we can formulate the following:

***Theorem* 3.1.2 (Buslaev).** *Assume that* (1.2) *takes place and* $\mathfrak{f}$ *is a non-trivial solution of* (1.1)*. Then* $R_0(\mathfrak{f})$ *is equal to the absolute value of one of the zeros of* α*, the coefficients of* $\mathfrak{f}$ *satisfy a reduced recurrence relation of the form*

$$f_n + \beta_{n,1} f_{n-1} + \cdots + \beta_{n,\ell} f_{n-\ell} = 0, \qquad \lim_n \beta_{n,k} = b_k, \qquad k = 1, \ldots, \ell, \tag{3.2}$$

where $\ell \leq m$ *is equal to the number of zeros of* α *on the circle* $\{z : |z| = R_0(\mathfrak{f})\}$*, all the zeros of* $\beta(z) = 1 + b_1 z + \cdots b_\ell z^\ell$ *lie on that circle,* β *divides* α*, and at least one of its zeros is a singular point of* $\mathfrak{f}$*.*

In [3, Theorem 1] it is required that the solution of (1.1) under consideration is not a polynomial. We have stipulated that $\alpha_{n,m} \neq 0, n \geq m$. This restriction implies that any non-trivial solution cannot be a polynomial. In fact, the contrary would imply that $f_n = 0$ for all $n \geq n_0$ and solving the recurrence backwards (taking advantage that $\alpha_{n,m} \neq 0, n \geq m$,) we conclude that $f_n = 0, n \geq 0$, and the solution would be the trivial one.

In [3, Theorem 2] the author imposes that $0 < R_0(\mathfrak{f}) < \infty$. Our assumptions imply this as Theorem 3.1.1 shows. The polynomials that we have denoted α_n play the role of the functions α_n in [3, Theorem 2].

When all the zeros of α have distinct absolute value, it is easy to see that Buslaev's theorem reduces to Poincare's result. A natural question arises.

Question: Is there a fundamental system of solutions of (1.1) such that every zero of α is a singular point of at least one function in the system?

The answer to this question is positive if it is known in advance that (1.1) has a fundamental system of solutions of the form $(\mathfrak{f}, z\mathfrak{f}, \ldots, z^{m-1}\mathfrak{f})$ for some (formal) Taylor expansion $\mathfrak{f}$ about the origin. This is a consequence of Theorem 1.1.6 proved by S.P. Suetin in [34, Theorem 1].

§ 3.2. Statement of the main result.

Without loss of generality, let the zeros of α be enumerated in such a way that

$$0 < |\zeta_1| \leq |\zeta_2| \leq \cdots \leq |\zeta_m|. \tag{3.3}$$

If several ζ_k coincide, they are enumerated consecutively. We shall also assume that the zeros in the collections of points $\mathcal{P}_n$ are indexed so that

$$\lim_{n\to\infty} \zeta_{n,k} = \zeta_k, \qquad k = 1, \ldots, m. \tag{3.4}$$

Several circles centered at the origin may contain more than one zero of α (or a zero of α of multiplicity greater than 1). Let $\mathcal{C}$ be one such circle (if any). Let $\zeta_j, \zeta_{j+1}, \ldots, \zeta_{j+N-1}$ be the zeros of α lying on $\mathcal{C}$. In this case, we assume that

$$\limsup_{n\to\infty} |\zeta_{n,k} - \zeta_k|^{1/n} < 1, \qquad k = j, \ldots, j+N-1. \tag{3.5}$$

The existence of such a circle $\mathcal{C}$ is not required. If such a circle does not exist, all the zeros of α have distinct absolute value and we are in the situation of the Poincaré and Perron theorems.

***Theorem* 3.2.1.** *Assume that* (3.4) *and* (3.5) *take place. Then there is a fundamental system of solutions* $(\mathfrak{f}^1, \ldots, \mathfrak{f}^m)$ *of* (1.1) *such that* $R_0(\mathfrak{f}^k) = |\zeta_k|, k = 1, \ldots, m,$ *and* ζ_k *is a singular point of* $\mathfrak{f}^k$*. Each* ζ_k *verifying* (3.5) *is a pole of* $\mathfrak{f}^k$*. Moreover, if* ζ_k *is a zero of multiplicity* τ *and* $\zeta_k = \zeta_{k+1} = \cdots = \zeta_{k+\tau-1}$ *then for each* $s = 1, \ldots, \tau$*,* $\mathfrak{f}^{k+s-1}$ *is analytic in a disk of radius larger than* $|\zeta_k|$ *except for a pole of exact order* s *at* ζ_k*.*

If the zeros of α have distinct absolute value, the statement of Theorem 3.2.1 is deduced directly from the Perron and Fabry theorems. When (3.5) takes place for $k = 1, \ldots, m$, the thesis of Theorem 3.2.1 follows directly from the fact that statement (b) of [10, Theorem 1.4] implies statement (a) of the same theorem.

We will see some consequences of Theorem 3.2.1 in the study of row sequences of Hermite-Padé approximation. We will not state those results here to avoid introducing more concepts and notation. In the next Section, we prove some auxiliary results needed for the proof of Theorem 3.2.1.

§ 3.3. Some auxiliary lemmas.

The proof of Theorem 3.2.1 is somewhat constructive. We start out from a fundamental system of solutions of (1.1) and through analytic continuation, carried out in successive steps, we find another fundamental system of solutions which fulfills the desired properties. As we carry out these steps, we find collections of solutions which according to Buslaev's theorem have radius of convergence equal to the absolute value of a zero of α. On any such circle, there may fall one or several zeros of α. The proof distinguishes two cases. The first when all the zeros on the circle satisfy (3.5). In this case, the analytic continuation is based on [3, Corollary 2]. If the circle contains only one zero of α of multiplicity 1 and we do not have (3.5), we adapt a proof of Perron's Theorem given by M.A. Evgrafov in [12] to continue the process.

Lemma 3.3.1 is exactly [3, Corollary 2]. We state it for the reader's convenience and refer to the original paper for the proof.

***Lemma* 3.3.1.** *Suppose that the assumptions of Buslaev's theorem hold and* $\mathfrak{f}$ *is a non trivial solution of* (1.1). *Let* $\zeta_j, \ldots, \zeta_{j+N-1}, N \geq 1$, *be the zeros of* α *on the circle* $\{z : |z| = R_0(\mathfrak{f})\}$ *and suppose that* (3.5) *takes place. Then* $R_0(\mathfrak{g}) > R_0(\mathfrak{f})$, *where* $\mathfrak{g}(z) = \prod_{k=j}^{j+N-1}(z - \zeta_k)\mathfrak{f}(z)$.

Evgrafov's proof of Perron's theorem is based on several lemmas. His paper has not been translated so for completeness we include the proof of those results that we will need. In this regard, Lemmas 3.3.2 and 3.3.3 correspond to [12, Lemma 1] and [12, Lemma 2], respectively, while Lemma 3.3.4 is a slightly improved version of [12, Lemma 3], which is necessary to cover our more general situation.

***Lemma* 3.3.2.** *Let* $(f_n)_{n\geq 0}$ *be a solution of* (1.1) *and let* $(\gamma_n)_{n\geq 0}$ *be a sequence such that* $\gamma_n \neq 0, n \geq 0$. *Then* $(F_n = f_n/\gamma_n)_{n\geq 0}$ *is a solution of the recurrence relation*

$$F_n + \alpha'_{n,1}F_{n-1} + \cdots + \alpha'_{n,m}F_{n-m} = 0, \qquad n \geq m, \tag{3.6}$$

where

$$\alpha'_{n,j} = \alpha_{n,j}\frac{\gamma_{n-j}}{\gamma_n}, \qquad j = 1, \ldots, m.$$

Moreover, if $\lim_{n\to\infty} \gamma_{n+1}/\gamma_n = 1$ *and the* $\alpha_{n,j}, j = 1, \ldots, m$, *verify* (1.2) *then so do the* $\alpha'_{n,j}, j = 1, \ldots, m$, *and the recurrences* (1.1) *and* (3.6) *have the same characteristic polynomial.*

Proof. Indeed, if we divide (1.1) by γ_n, we obtain that for all $n \geq m$

$$0 = \frac{f_n}{\gamma_n} + \alpha_{n,1}\frac{\gamma_{n-1}}{\gamma_n}\frac{f_{n-1}}{\gamma_{n-1}} + \cdots + \alpha_{n,m}\frac{\gamma_{n-m}}{\gamma_n}\frac{f_{n-m}}{\gamma_{n-m}},$$

which is (3.6).

Now, $\lim_{n\to\infty} \gamma_{n+1}/\gamma_n = 1$ implies that $\lim_{n\to\infty} \gamma_{n-j}/\gamma_n = 1, j = 1, \ldots, m$, so from (1.2) it follows that

$$\lim_{n\to\infty} \alpha'_{n,j} = \lim_{n\to\infty} \alpha_{n,j} = a_j, \qquad j = 1, \ldots, m.$$

Therefore, in that case both equations have the same characteristic polynomial. ■

***Proposition* 3.3.1.** *The sequences* $(f_n^j)_{n\geq 0}$, $j = 1, \ldots, N, 1 \leq N \leq m$, *are linearly independent solutions of* (1.1) *if and only if* $(f_n^j/\gamma_n)_{n\geq 0}$, $j = 1, \ldots, N$, *(where* $\gamma_n \neq 0$, $n \geq 0$*) are linearly independent solutions of* (3.6).

Proof. Non-zero vectors are linearly independent so the statement is meaningful for $N \geq 2$. From the definition the sequences $(f_n^j)_{n\geq 0}$, $j = 1, \ldots, N$, are linearly dependent if and only if there exist constants $\alpha_1, \ldots, \alpha_N$, not all zero, such that

$$\alpha_1 \left(f_n^1\right)_{n\geq 0} + \cdots + \alpha_N \left(f_n^N\right)_{n\geq 0} = \mathbf{0},$$

where $\mathbf{0}$ denotes the null vector. Obviously, this occurs if and only if

$$\alpha_1 f_n^1 + \cdots + \alpha_N f_n^N = 0, \qquad n \geq 0,$$

and since $\gamma_n \neq 0$ for all n this happens if and only if

$$\alpha_1 \frac{f_n^1}{\gamma_n} + \cdots + \alpha_N \frac{f_n^N}{\gamma_n} = 0, \qquad n \geq 0,$$

or what is the same if and only if

$$\alpha_1 \left(\frac{f_n^1}{\gamma_n}\right)_{n\geq 0} + \cdots + \alpha_N \left(\frac{f_n^N}{\gamma_n}\right)_{n\geq 0} = \mathbf{0},$$

with constants $\alpha_1, \cdots, \alpha_N$, not all zero, which mean if and only if the sequences $(f_n^j/\gamma_n)_{n\geq 0}$, $j = 1, \ldots, N$ are linearly dependent. ■

***Lemma* 3.3.3.** *If equation* (1.1) *has the solution* $(\lambda^n)_{n\geq 0}, \lambda \neq 0$, *then its left hand side may be expressed in the form*

$$F_{n-1} + \beta_{n,1}F_{n-2} + \cdots + \beta_{n,m-1}F_{n-m} = 0, \tag{3.7}$$

where

$$F_n = f_{n+1} - \lambda f_n.$$

Additionally, if the $\alpha_{n,j}$ *verify* (1.2) *then*

$$\lim_{n,j} \beta_{n,j} = b_j, \qquad j = 1, \ldots, m-1, \qquad b_{m-1} \neq 0, \tag{3.8}$$

and the characteristic polynomials of (1.1) *and* (3.7) *are connected by the relation*

$$(z-\lambda)(z^{m-1} + b_1 z^{m-2} + \cdots + b_{m-1}) = z^m + a_1 z^{m-1} + \cdots + a_m. \tag{3.9}$$

Proof. Consider the polynomial

$$p_n(z) = z^m + \alpha_{n,1}z^{m-1} + \cdots + \alpha_{n,m}.$$

Substituting the solution $(\lambda^n)_{n\geq 0}$ in (1.1) and factoring out λ^{n-m}, we get

$$p_n(\lambda) = \lambda^m + \alpha_{n,1}\lambda^{m-1} + \cdots + \alpha_{n,m} = 0;$$

therefore,

$$\frac{p_n(z)}{z-\lambda} = z^{m-1} + \beta_{n,1}z^{m-2} + \cdots + \beta_{n,m-1}.$$

That is

$$p_n(z) = (z-\lambda)(z^{m-1} + \beta_{n,1}z^{m-2} + \cdots + \beta_{n,m-1}). \tag{3.10}$$

Equating coefficients of equal power of z, we obtain

$$\beta_{n,j} - \lambda\beta_{n,j-1} = \alpha_{n,j}, \qquad j = 1, \ldots, m, \qquad \beta_{n,0} = 1, \qquad \beta_{n,m} = 0. \tag{3.11}$$

In particular,

$$\beta_{n,m-1} = -\alpha_{n,m}/\lambda \neq 0.$$

From (3.11) the existence of $\lim_{n\to\infty} \alpha_{n,j}$ and $\lim_{n\to\infty} \beta_{n,j-1}$ imply the existence of $\lim_{n\to\infty} \beta_{n,j}$. Since $\beta_{n,0} = 1$, it follows that (1.2) implies (3.8) and then (3.9) is immediate taking limit over n in (3.10).

It remains to verify (3.7). Put $F_n = f_{n+1} - \lambda f_n$ in the left hand side of (3.7). Using (3.11) and (1.1), we get

$$F_{n-1} + \beta_{n,1}F_{n-2} + \cdots + \beta_{n,m-1}F_{n-m} =$$
$$f_n - \lambda f_{n-1} + \beta_{n,1}(f_{n-1} - \lambda f_{n-2}) + \cdots + \beta_{n,m-1}(f_{n-m+1} - \lambda f_{n-m}) =$$
$$f_n + (\beta_{n,1} - \lambda\beta_{n,0})f_{n-1} + \cdots + (\beta_{n,m} - \lambda\beta_{n,m-1})f_{n-m} = 0$$

as we needed to prove. ■

***Proposition* 3.3.2.** *Suppose that* $(\lambda^n)_{n\geq 0}, \lambda \neq 0$, *is a solution of* (1.1). *Then* $(f_n^k)_{n\geq 0}$, $k = 1, \ldots, N \leq m-1$, *and* $(\lambda^n)_{n\geq 0}$ *constitute a system of linearly independent solutions of* (1.1) *if and only if* $(F_n^k)_{n\geq 0}$, $k = 1, \ldots, N$, *is a system of linearly independent solutions of* (3.7), *where* $F_n^k = f_{n+1}^k - \lambda f_n^k, k = 1, \ldots, N$.

Proof. Set $f_n^0 = \lambda^n$, $n \geq 0$. Assume that $\left(f_n^k\right)_{n\geq 0}$, $k = 0, \ldots, N$ are linearly dependent solutions of (1.1). Then, there exist constants $\alpha_0, \ldots, \alpha_N$, not all zero, such that

$$\sum_{k=0}^{N} \alpha_k f_n^k = 0, \qquad n \geq 0. \tag{3.12}$$

Multiplying by λ we get

$$\sum_{k=0}^{N} \alpha_k \lambda f_n^k = 0, \qquad n \geq 0. \tag{3.13}$$

Deleting (3.13) for the index n from (3.12) for the index $n+1$ we obtain

$$\begin{aligned} 0 &= \sum_{k=0}^{N} \alpha_k \left(f_{n+1}^k - \lambda f_n^k\right) = \sum_{k=1}^{N} \alpha_k \left(f_{n+1}^k - \lambda f_n^k\right) \\ &= \sum_{k=1}^{N} \alpha_k F_n^k, \quad n \geq 0, \end{aligned} \tag{3.14}$$

because $f_{n+1}^0 - \lambda f_n^0 = \lambda^{n+1} - \lambda^{n+1} = 0$ and for $k = 1, \ldots, N$ $F_n^k = f_{n+1}^k - f_n^k$.

Now if $\alpha_0 = 0$, then one of the coefficients $\alpha_1, \ldots, \alpha_N$ must be different from zero because of the definition of linear dependence. On the other hand, if $\alpha_0 \neq 0$ then also one of the coeffients $\alpha_1, \ldots, \alpha_N$ must differ from zero since otherwise (3.12) could not hold. We have shown that these exist coefficients $\alpha_1, \ldots, \alpha_N$, not all zero such (3.14) takes place. Consequently, the system $\left(F_n^k\right)_{n\geq 0}$, $k = 1, \ldots, N$ is linearly dependent.

Reciprocally, suppose that the system $\left(F_n^k\right)_{n\geq 0}$, $k = 1, \ldots, N$ is linearly dependent. Then, there exist constants $\alpha_1, \ldots, \alpha_N$ not all zero such that

$$0 = \sum_{k=1}^{N} \alpha_k F_n^k = \sum_{k=1}^{N} \alpha_k f_{n+1}^k - \lambda \sum_{k=1}^{N} \alpha_k f_n^k, \quad n \geq 0. \tag{3.15}$$

If we denote $f_n = \sum_{k=1}^{N} \alpha_k f_n^k$ relation (3.15) entails that

$$f_{n+1} = \lambda f_n, \qquad n \geq 0. \tag{3.16}$$

Applying (3.16) recurrently we find that

$$f_n = \lambda f_{n-1} = \lambda^2 f_{n-2} = \lambda^n f_0, \qquad n \geq 0. \tag{3.17}$$

Suppose that $f_0 = 0$, then $f_n = 0$, $n \geq 0$, and it follows that

$$\sum_{k=1}^{N} \alpha_k f_n^k = 0, \qquad n \geq 0$$

which means that the system $\left(f_n^k\right)_{n\geq 0}$, $k = 1, \ldots, N$ is linearly dependent. If $f_0 \neq 0$ then relation (3.17) tells us that

$$(f_n)_{n\geq 0} = f_0 \left(f_n^0\right)_{n\geq 0},$$

therefore

$$f_n^0 = \sum_{k=1}^{N} \frac{\alpha_k}{f_n^0} f_n^k, \qquad n \geq 0$$

which implies that the system $\left(f_n^k\right)_{n\geq 0}$, $k = 0, \ldots, N$ is linearly dependent. With this we conclude the proof. ■

***Lemma* 3.3.4.** *Suppose that* $(F_n)_{n\geq 0}$ *is such that* $\limsup_{n\to\infty} |F_n|^{1/n} = \mu, \mu \neq |\lambda|$. *Then, there exists a solution* $(f_n)_{n\geq 0}$ *of the equations* $F_n = f_{n+1} - \lambda f_n$, $n \geq 0$, *such that* $\limsup_{n\to\infty} |f_n|^{1/n} = \mu$.

Proof. We give two different expression for the solution $(f_n)_{n\geq 0}$ depending on whether $\mu < |\lambda|$ or $\mu > |\lambda|$. In the first case we set

$$f_n = -\frac{F_n}{\lambda} - \frac{F_{n+1}}{\lambda^2} - \frac{F_{n+2}}{\lambda^3} - \cdots, \tag{3.18}$$

and in the second

$$f_n = F_{n-1} + \lambda F_{n-2} + \cdots + \lambda^{n-1} F_0. \tag{3.19}$$

We will see in a minute that (3.18) is convergent for each n. With this in mind, it is easy to verify that so defined the sequence $(f_n)_{n\geq 0}$ satisfies the required equations.

Let us verify that the numbers f_n in (3.18) are finite and $\limsup_{n\to\infty} |f_n|^{1/n} = \mu$. Indeed, take $\varepsilon > 0$ such that $\mu + \varepsilon < |\lambda|$. From the assumption on the F_n we get that there exists some constant $C \geq 1$ such that

$$|F_n| \leq C(\mu + \varepsilon)^n, \qquad n \geq 0.$$

Consequently,

$$|f_n| \le C\left(\frac{(\mu+\varepsilon)^n}{|\lambda|} + \frac{(\mu+\varepsilon)^{n+1}}{|\lambda|^2} + \frac{(\mu+\varepsilon)^{n+2}}{|\lambda|^3} + \cdots\right) = \frac{C(\mu+\varepsilon)^n}{|\lambda| - (\mu+\varepsilon)} < \infty. \tag{3.20}$$

Moreover, (3.20) implies that

$$\limsup_{n\to\infty} |f_n|^{1/n} \le \mu + \varepsilon.$$

Letting ε tend to zero we get that $\limsup_{n\to\infty} |f_n|^{1/n} \le \mu$. On the other hand, $|F_n| \le |f_{n+1}| + |\lambda f_n|$. This in turn implies that $\mu = \limsup_{n\to\infty} |F_n|^{1/n} \le \limsup_{n\to\infty} |f_n|^{1/n}$, so indeed we have equality.

When $\mu > |\lambda|$, we proceed analogously. Using (3.19), we have

$$|f_n| \le C(\mu+\varepsilon)^{n-1}\left(1 + \frac{|\lambda|}{\mu+\varepsilon} + \cdots + \frac{|\lambda|^{n-1}}{(\mu+\varepsilon)^{n-1}}\right) \le \frac{C(\mu+\varepsilon)^n}{\mu+\varepsilon-|\lambda|}. \tag{3.21}$$

From (3.21) we get that $\limsup_{n\to\infty} |f_n|^{1/n} \le \mu$, and equality is derived just as before. ∎

§ 3.4. Proof of the main result.

Proof. Let $\mathbf{f}^1 = (\mathfrak{f}^{1,1}, \mathfrak{f}^{1,2}, \ldots, \mathfrak{f}^{1,m})$ be a fundamental system of solutions of (1.1) and $\zeta_1, \ldots, \zeta_m$ the collection of zeros of α enumerated as in (3.3). According to Buslaev's theorem

$$0 < |\zeta_1| \le R_0(\mathfrak{f}^{1,j}) \le |\zeta_m| < \infty, \qquad j = 1, \ldots, m.$$

Let $D_0(\mathbf{f}^1)$ denote the intersection of all the disks centered at the origin and radii $R_0(\mathfrak{f}^{1,j}), j = 1, \ldots, m$ whose boundary we denote $\mathcal{C}_1$. By Buslaev's theorem, the circle $\mathcal{C}_1$ of radius $R_1 := R_0(\mathbf{f}^1)$ contains at least one zero of α. Moreover, several (and at least one) of the functions in $\mathbf{f}^1$ has radius of convergence equal to $R_1 = R_0(\mathbf{f}^1)$. Let $\zeta_j, \ldots, \zeta_{j+N-1}$ be the collection of all the zeros of α lying on $\mathcal{C}_1$. We distinguish the cases when $N \ge 2$ and when $N = 1$. We remark that so far we cannot assert that $|\zeta_j| = |\zeta_1|$; in principle, it may be larger.

Suppose that $N \ge 2$. In this case, according to our assumptions, (3.5) takes place. Due to Lemma 3.3.1, $R_0(\prod_{k=j}^{j+N-1}(z-\zeta_k)\mathfrak{f}^{1,\ell}) > R_0(\mathbf{f}^1) = R_1, \ell = 1, \ldots, m$; that is, either $\mathfrak{f}^{1,\ell}$ has radius of convergence larger than R_1 to start with or, $\mathfrak{f}^{1,\ell}$ has at most poles on $\mathcal{C}_1$ at zeros of α and their order is less than or equal to the multiplicity of the corresponding zero of α.

Let us find coefficients $c_1, \dots, c_m$ such that

$$\sum_{\ell=1}^{m} c_\ell \mathfrak{f}^{1,\ell} \tag{3.22}$$

is analytic in a neighborhood of $\overline{D_0(\mathbf{f}^1)}$. Finding the coefficients c_ℓ reduces to solving a homogeneous linear system of N equations on m unknowns. In fact, if ζ is one of the zeros of α on $\mathcal{C}_1$ and it has multiplicity τ we obtain τ equations choosing the coefficients c_ℓ so that

$$\int_{|\omega-\zeta|=\delta} (\omega-\zeta)^\nu \left(\sum_{\ell=1}^{\mathbf{m}} c_\ell \mathfrak{f}^{1,\ell}(\omega) \right) d\omega = 0, \qquad \nu = 0, \dots, \tau-1. \tag{3.23}$$

where δ is sufficiently small. We do the same with each distinct zero of α on $\mathcal{C}_1$. The homogeneous linear system of N equations so obtained has $m - N_1, N_1 \leq N$, linearly independent solutions, where N_1 equals the rank of the linear system of equations. Denote the solutions of the linear system by $\mathbf{c}^{1,j}$, $j = N_1+1, \dots, m$. Set

$$\mathbf{c}^{1,j} = (c_1^{1,j}, \dots, c_m^{1,j}), \qquad j = N_1+1, \dots, m,$$

and

$$\mathfrak{f}^{2,j} = \sum_{\nu=1}^{m} c_\nu^{1,j} \mathfrak{f}^{1,\nu}, \qquad j = N_1+1, \dots, m.$$

We wish to emphasize several points:

1. The collection of functions $\mathbf{f}^2 = (\mathfrak{f}^{2,N_1+1}, \dots, \mathfrak{f}^{2,m})$ is made up of non-trivial linearly independent solutions of (1.1).

2. Because of (3.23), $R_0(\mathfrak{f}^{2,j}) > R_1, j = N_1+1, \dots, m$.

3. If $N_1 = N$; that is, the system of equations has full rank, then the system is solvable if for some specific value of $\nu = 0, \dots, \tau-1$ in (3.23) instead of equating the left hand side to zero we equate it to 1. Doing this for each zero ζ of α on $\mathcal{C}_1$ and for each $\nu = 0, \dots, \tau-1$, we obtain N_1 linearly independent solutions of (1.1) which are meromorphic on a neighborhood of $\overline{D_0(\mathbf{f}^1)}$ except for a pole of exact order $\nu+1$ at ζ. On this circle, this would settle the last statement of the theorem. (We will show that on each circle containing more than one zero of α the corresponding system of equations has full rank. This conclusion will be drawn at the very end of the proof of the theorem.)

Now let us suppose that $\mathcal{C}_1$ contains only one zero ζ_j of α of multiplicity 1; that is $N = 1$. If, nevertheless, (3.5) takes place we could proceed as before, so we will not use (3.5) in the arguments that follow. We have,

$$R_0(\mathfrak{f}^{1,\nu}) \geq R_1 = R_0(\mathbf{f}^1) = |\zeta_j|, \qquad \nu = 1, \ldots, m,$$

with equality for some ν. Without loss of generality we can assume

$$R_0(\mathfrak{f}^{1,1}) = \cdots = R_0(\mathfrak{f}^{1,M}) = R_1, \qquad 1 \leq M \leq m,$$

and $R_0(\mathfrak{f}^{1,\nu}) > R_1, \nu = M+1, \ldots, m$ (if any). According to (3.2) (with $\ell = 1$) and the fact that ζ_j is the unique zero of α on $\mathcal{C}_1$, we have

$$\lim_{n\to\infty} \frac{f_{n-1}^{1,\nu}}{f_n^{1,\nu}} = \zeta_j, \qquad \nu = 1, \ldots, M, \tag{3.24}$$

where $(f_n^{1,\nu})_{n\geq 0}$ denotes the collection of Taylor coefficients of $\mathfrak{f}^{1,\nu}$, and ζ_j is a singular point of each $\mathfrak{f}^{1,\nu}, \nu = 1, \ldots, M$. Should $M = 1$ we have obtained one solution of (1.1), namely $\mathfrak{f}^{1,1}$, with radius of convergence R_1 and the remaining solutions in $\mathbf{f}^1$ have radius of convergence larger than R_1. We aim to show that if $M > 1$ we can also find one solution of (1.1) with radius R_1 and additional $m - 1$ linearly independent solutions of (1.1) (not necessarily $\mathfrak{f}^{1,2}, \ldots, \mathfrak{f}^{1,m}$) with radius larger than R_1.

Without loss of generality, we can assume that $f_n^{1,1} \neq 0, n \geq 0$. Indeed, (3.24) entails that $f_n^{1,1} \neq 0, n \geq n_0$. Let $T_n(\mathfrak{f})$ be the Taylor polynomial of $\mathfrak{f}$ of degree n. Consider the collection of functions $\hat{\mathfrak{f}}^1, \ldots, \hat{\mathfrak{f}}^M$, where

$$\hat{\mathfrak{f}}^\nu(z) = (\mathfrak{f}^{1,\nu}(z) - T_{n_0}(\mathfrak{f}^{1,\nu})(z))/z^{n_0}, \qquad \nu = 1, \ldots, M.$$

It is easy to verify that these functions are linearly independent, satisfy the recurrence (1.1) with the indices n shifted by n_0, have radii of convergence equal to R_1, have the same singularities as the corresponding $\mathfrak{f}^{1,\nu}$ on $\mathcal{C}_1$, and $\hat{f}_n^\nu \neq 0, n \geq 0$. Should it be necessary, we derive the desired properties of the functions $\mathfrak{f}^{1,1}, \ldots, \mathfrak{f}^{1,M}$ from $\hat{\mathfrak{f}}^1, \ldots, \hat{\mathfrak{f}}^M$.

Let $\lambda = \zeta_j^{-1}$. This point is a root of the characteristic polynomial $p(z) = z^m\alpha(1/z)$ of (1.1). Set

$$\gamma_n = f_n^{1,1}/\lambda^n, \qquad n \geq 0.$$

According to Lemma 3.3.2, $(\lambda^n)_{n\geq 0}, \lambda^n = f_n^{1,1}/\gamma_n$, is a solution of (3.6). From (3.24) we have

$$\lim_{n\to\infty} \frac{\gamma_{n+1}}{\gamma_n} = \frac{1}{\lambda} \lim_{n\to\infty} \frac{f_{n+1}^{1,1}}{f_n^{1,1}} = 1. \tag{3.25}$$

Consequently, the recurrences (1.1) and (3.6) have the same characteristic polynomial. Aside from the solution $(\lambda^n)_{n\geq 0}$, (3.6) also has the solutions $(f_n^{1,\nu}/\gamma_n)_{n\geq 0}, \nu = 2, \ldots, M$. Using Lemma 3.3.3, with these solutions of (3.6) we can construct $M-1$ linearly independent solutions $(F_n^\nu)_{n\geq 0}, \nu = 2, \ldots, M$, of (3.7) where

$$F_n^\nu = \frac{f_{n+1}^{1,\nu}}{\gamma_{n+1}} - \lambda \frac{f_n^{1,\nu}}{\gamma_n}, \qquad n \geq 0, \qquad \nu = 2, \ldots, M. \tag{3.26}$$

The radii of convergence of the functions $\mathfrak{f}^{1,\nu}, \nu = 2, \ldots, M$, equal $R_1 = |\lambda|^{-1}$. This together with (3.25) and (3.26) imply that

$$\limsup_{n\to\infty} |F_n^\nu|^{1/n} \leq R_1^{-1}, \qquad \nu = 2, \ldots, M.$$

According to Buslaev's theorem, for each $\nu = 2, \ldots, M$ the radius of convergence of $F^\nu(z) = \sum_{n=0}^\infty F_n^\nu z^n$ is equal to the reciprocal of the absolute value of one of the zeros of the characteristic polynomial $\hat{p}(z) = z^{m-1} + \beta_1 z^{m-1} + \cdots + \beta_{m-1}$ associated with (3.7). According to (3.9), and our assumptions, λ is not a zero of $\hat{p}$. Therefore,

$$\limsup_{n\to\infty} |F_n^\nu|^{1/n} < R_1^{-1}, \qquad \nu = 2, \ldots, M.$$

Using Lemma 3.3.4 and Lemma 3.3.3, with formula (3.18) we can find $M-1$ linearly independent solutions $\hat{\mathfrak{f}}^\nu, \nu = 2, \ldots, M$, of the recurrence (3.7) whose radius of convergence is greater than R_1. Now, using again Lemma 3.3.2 with $\gamma_n = \lambda^n/f_n^{1,1}, n \geq 0$ we find a system $\mathbf{f}^2 = (\mathfrak{f}^{2,2}, \ldots, \mathfrak{f}^{2,m})$ of $m-1$ linearly independent solutions of (1.1) each of which has radius of convergence greater than $R_1 = |\zeta_j|$. Here, $(\mathfrak{f}^{2,2}, \ldots, \mathfrak{f}^{2,M})$ comes from the last application of Lemma 3.3.2 whereas $\mathfrak{f}^{2,\nu} = \mathfrak{f}^{1,\nu}, \nu = M+1, \ldots, m$. Summarizing, we have found that $R_0(\mathfrak{f}^{1,1}) = |\zeta_j|$ with ζ_j a singular point of $\mathfrak{f}^{1,1}$ and all the functions in $\mathbf{f}^2$ are linearly independent solutions of (1.1) with radius of convergence larger than $|\zeta_j|$.

Now we proceed with $\mathbf{f}^2$ exactly the same way as we did with $\mathbf{f}^1$ and construct a system $\mathbf{f}^3$ with $m - N_1 - N_2$ linearly independent solutions of (1.1), where N_2 denotes either the rank of the corresponding system of homogeneous linear equations (when the circle $\mathcal{C}_2$ has more than one zero of α) or $N_2 = 1$ if $\mathcal{C}_2$ has exactly one zero of α. In a finite number of steps, say r, we either run out of linearly independent solutions of (1.1) because $N_1 + \ldots + N_r = m$ or we find at least one non trivial solution of (1.1) with radius of convergence $R > |\zeta_m|$.

The second possibility is impossible because according to Buslaev's theorem R has to be equal to the absolute value of one of the zeros of α. On the other hand, since $N_k, k = 1, \ldots, r$, is less than or equal to the number of zeros, say $\hat{N}_k$, of α on $\mathcal{C}_k$, if $N_1 + \ldots + N_r = m$, if follows that $N_k = \hat{N}_k, k = 1, \ldots, r$. This happens only when $\cup_{k=1}^r \mathcal{C}_k$ contains all the zeros of α (we skip no circle at all as we carry out the process). In turn this means that either any given circle contains exactly one zero of α or the rank of the corresponding homogeneous linear system of equations is equal to the number of zeros of α on the circle. As we saw in the proof of the first step this means that associated with each circle $\mathcal{C}_k$ we have N_k linearly independent solutions of (1.1) with the properties announced in the statement of the theorem. Since the circles are contained one inside the other the collection of these linearly independent solutions form a fundamental system of solutions of (1.1). ■

§ 3.5. Consequences for Hermite-Padé approximation.

Let $\mathbf{f} = (\mathfrak{f}^1, \ldots, \mathfrak{f}^d)$ and $\mathbf{m} = (m_1, \ldots, m_d) \in \mathbb{Z}^d_{\geq 0} \setminus \{\mathbf{0}\}$ be fixed and let $q_{n,\mathbf{m}}$ denote the denominator of the $(n, \mathbf{m})$ Hermite Padé approximation of $\mathbf{f}$ with respect to $n \geq \max\{m_1, \ldots, m_d\}$. Set

$$q_{n,\mathbf{m}}(z) = b_{n,|\mathbf{m}|} z^{|\mathbf{m}|} + b_{n,|\mathbf{m}|-1} z^{|\mathbf{m}|-1} + \cdots + b_{n,0}, \tag{3.27}$$

where $b_{n,0} = 1$ if $q_{n,0}(0) \neq 0$ and equals 0 otherwise (see Definition 1.2.1). It would be more appropriate to write $b_{n,\mathbf{m},k}$ in place of $b_{n,k}, k = 0, \ldots, |\mathbf{m}|$, but since $\mathbf{m}$ will remain fixed we drop it to shorten the notation.

In the sequel, we will assume that the sequence (3.27), $n \geq \max\{m_1, \ldots, m_d\}$, verifies

$$\lim_{n \longrightarrow \infty} q_{n,\mathbf{m}}(z) = q_{\mathbf{m}}(z) = \prod_{k=1}^{|\mathbf{m}|} (1 - z\zeta_k^{-1}), \quad \deg q_{\mathbf{m}} = |\mathbf{m}|, \quad q_{\mathbf{m}}(0) = 1. \tag{3.28}$$

***Theorem* 3.5.1.** *Let* $\mathbf{m} = (m_1, \ldots, m_d) \in \mathbb{N}^d$ *be given. Suppose that* $\mathbf{f} = (\mathfrak{f}^1, \ldots, \mathfrak{f}^d)$ *is a vector of formal power expansions as in* (1.14) *which is polynomially independent with respect to* $\mathbf{m}$. *Assume that* (3.3), (3.4), *and* (3.5) *take place. Then, each* $\zeta_k, k = 1, \ldots, |\mathbf{m}|$, *is a system singularity of* $(\mathbf{f}, \mathbf{m})$. *Moreover, if* ζ_k *is a zero of multiplicity* τ_k *of* $q_{\mathbf{m}}$ *which verifies* (3.5), *then it is a system pole of* $(\mathbf{f}, \mathbf{m})$ *of order* τ_k.

Proof. We have $\lim_{n\to\infty} q_{n,\mathbf{m}} = q_{\mathbf{m}}$ where $\deg q_{\mathbf{m}} = |\mathbf{m}|$ and $q_{\mathbf{m}}(0) = 1$. Therefore, there exists n_0 such that $b_{n,|\mathbf{m}|} \neq 0, b_{n,0} = 1$, for all $n \geq n_0$.

Given $\mathfrak{f}(z) = \sum_{\nu=0}^\infty f_\nu z^\nu$ define

$$\hat{\mathfrak{f}}(z) := (\mathfrak{f}(z) - T_{n_0}(\mathfrak{f})(z))/z^{n_0} = \sum_{\nu=0}^\infty f_{n_0+\nu} z^\nu = \sum_{\nu=0}^\infty \hat{f}_\nu z^\nu,$$

where $T_{n_0}(\mathfrak{f})$ is the Taylor polynomial of $\mathfrak{f}$ of degree $n_0 - 1$. Notice that the coefficients of $\hat{\mathfrak{f}}$ are shifted by n_0 in relation with the coefficients of $\mathfrak{f}$; that is

$$\hat{f}_\nu = f_{n_0+\nu}, \qquad \nu \geq 0.$$

Consequently,

$$[q_{n_0+n,\mathbf{m}}\mathfrak{f}]_{n_0+n} = f_{n+n_0} + b_{n+n_0,1}f_{n+n_0-1} + \cdots + b_{n+n_0,|\mathbf{m}|}f_{n+n_0-|\mathbf{m}|} =$$

$$\hat{f}_n + b_{n+n_0,1}\hat{f}_{n-1} + \cdots + b_{n+n_0,|\mathbf{m}|}\hat{f}_{n-|\mathbf{m}|}$$

with $b_{n+n_0,|\mathbf{m}|} \neq 0, n \geq 0$. If we set $\hat{q}_{n,\mathbf{m}} = q_{n+n_0,\mathbf{m}}$, we have

$$[q_{n_0+n,\mathbf{m}}\mathfrak{f}]_{n_0+n} = 0 \qquad \text{if and only if} \qquad [\hat{q}_{n,\mathbf{m}}\hat{\mathfrak{f}}]_n = 0.$$

Set $\hat{\mathbf{f}} = (\hat{\mathfrak{f}}_1, \ldots, \hat{\mathfrak{f}}_d)$. Notice that the system of functions

$$(\hat{\mathfrak{f}}_1, \ldots, z^{m_1-1}\hat{\mathfrak{f}}_1, \mathfrak{f}^2, \ldots, z^{m_d-1}\hat{\mathfrak{f}}_d) \tag{3.29}$$

is linearly independent for otherwise the system $\mathbf{f}$ would not be polynomially independent. Consequently, (3.29) constitutes a fundamental system of solutions of the recurrence relations

$$[\hat{q}_{n,\mathbf{m}}\hat{\mathfrak{f}}]_n = 0, \qquad n \geq |\mathbf{m}|, \tag{3.30}$$

which verifies $\deg(\hat{q}_{n,\mathbf{m}}) = |\mathbf{m}|, \hat{q}_{n,\mathbf{m}}(0) = 1, n \geq 0$, and

$$\lim_{n\to\infty} \hat{q}_{n,\mathbf{m}} = q_{\mathbf{m}}.$$

Therefore, the recurrence relation (3.30) is exactly like (1.1), including the assumption that the coefficient of $\hat{q}_{n,\mathbf{m}}$ which plays the role of $\alpha_{n,m}$ (the coefficient accompanying $z^{|\mathbf{m}|}$) is different from zero, and fulfills the assumptions of Theorem 3.2.1.

Applying Theorem 3.2.1 to $(\hat{\mathbf{f}}, \mathbf{m})$ we obtain that this pair fulfills the thesis of Theorem 3.5.1. However, it is easy to verify that $(\mathbf{f}, \mathbf{m})$ and $(\hat{\mathbf{f}}, \mathbf{m})$ have the same system singularities and the same system poles including their order. ■

Two immediate consequences of Theorem 3.5.1 which are worth singling out are the following.

***Corollary* 3.5.1.** *Suppose that $\mathbf{f} = (\mathfrak{f}^1, \ldots, \mathfrak{f}^d)$ is a vector of formal power expansions as in* (1.14) *which is polynomially independent with respect to $\mathbf{m} = (m_1, \ldots, m_d) \in \mathbb{N}^d$. Assume that all the zeros of $q_{\mathbf{m}}$ verify* (3.5). *Then, $(\mathbf{f}, \mathbf{m})$ has exactly $|\mathbf{m}|$ system poles which coincide with the zeros of $q_{\mathbf{m}}$, taking account of their order.*

Corollary 3.5.1 contains the inverse statement, (b) implies (a), in [10, Theorem 1.4].

***Corollary* 3.5.2.** *Suppose that* $\mathbf{f} = (\mathfrak{f}^1, \ldots, \mathfrak{f}^d)$ *is a vector of formal power expansions as in* (1.14) *which is polynomially independent with respect to* $\mathbf{m} = (m_1, \ldots, m_d) \in \mathbb{N}^d$. *Assume that all the zeros of* $q_{\mathbf{m}}$ *have distinct absolute value. Then, all the zeros of* $q_{\mathbf{m}}$ *are system singularities of* $(\mathbf{f}, \mathbf{m})$.

This corollary was suggested to hold in the sentence following [21, Corollary 5.2]. In [21] there are other results on Hermite-Padé approximation which complement those given here.

CHAPTER 4

Direct and inverse results for Multipoint Hermite-Padé approximants

§ 4.1. Necessary and sufficient conditions for convergence.

We shall consider a general interpolation scheme for constructing vector rational approximations to a given vector of analytic functions which generalizes the construction of the classical Hermite-Padé approximants.

Let E be a bounded continuum with connected complement in the complex plane $\mathbb{C}$. By $\mathcal{H}(E)$ we denote the space of all functions holomorphic in some neighborhood of E. Set

$$\mathcal{H}(E)^d := \{(\mathfrak{f}_1, \ldots, \mathfrak{f}_d) : \mathfrak{f} \in \mathcal{H}(E), j = 1, \ldots, d\}.$$

Let $\alpha \subset E$ be a table of points; more precisely, $\alpha = \{\alpha_{n,k}\}$, $k = 1, \ldots, n$, $n = 1, 2, \ldots$. We propose the following definition.

***Definition* 4.1.1.** *Let* $\mathbf{f} \in \mathcal{H}(E)^d$*. Fix a multi-index* $\mathbf{m} = (m_1, \ldots, m_d) \in \mathbb{N}^d$ *and* $n \in \mathbb{N}$*. Set* $|\mathbf{m}| = m_1 + \cdots + m_d$*. Then, there exist polynomials* $P_{n,\mathbf{m},k}$*,* $Q_{n,\mathbf{m}}$*,* $k = 1, \ldots, d$ *such that*

b.1) $\deg P_{n,\mathbf{m},k} \leq n - m_k$, $\deg Q_{n,\mathbf{m}} \leq |\mathbf{m}|$, $Q_{n,\mathbf{m}} \not\equiv 0$,

b.2) $(Q_{n,\mathbf{m}}\mathfrak{f}_k - P_{n,\mathbf{m},k})/a_{n+1} \in H(E)$,

where $a_n(z) = \prod_{k=1}^{n}(z - \alpha_{n,k})$*. The vector rational function*

$$\mathbf{R}_{n,\mathbf{m}} = (R_{n,\mathbf{m},1}, \ldots, R_{n,\mathbf{m},d}) = (P_{n,\mathbf{m},1}, \ldots, P_{n,\mathbf{m},d})/Q_{n,\mathbf{m}}$$

is called multipoint Hermite-Padé (MHP) approximant of $\mathbf{f}$ *with respect to* $\mathbf{m}$ *and* α.

This vector rational approximation, in general, is not uniquely determined. Hereafter, we assume that given $(n, \mathbf{m})$, one particular solution is taken. Without loss of generality we can assume that $Q_{n,\mathbf{m}}$ is a monic polynomial that has no common zero simultaneously with all $P_{n,\mathbf{m},k}$. In all what follows $\mathbf{m}$ remains fixed and $\{\mathbf{R}_{n,\mathbf{m}}\}_{n\in\mathbb{N}}$ is called a row sequence of MHP of $\mathbf{f}$ with respect to $\mathbf{m}$. Notice that the degree of the common denominators remains fixed as n varies.

MHP reduce to classical Hermite-Padé approximants when E is a disk about the origin and $a_n(z) = z^n$. There are not many papers dealing with the convergence properties of row sequences of HP approximation. Here, we generalize the results in [9] to MHP approximants. Extensions in other directions using expansions in orthogonal and Faber polynomials of the verctor function to produce the vector rational approximants of $\mathbf{f}$ were provided in [6, 7]. For other approaches to the study of row sequences of vector valued rational approximation see [32] and [36].

In analogue with Definition 1.2.2 we give

***Definition* 4.1.2.** *Given* $\mathbf{f} = (\mathfrak{f}_1, \ldots, \mathfrak{f}_d) \in \mathcal{H}(E)^d$ *and* $\mathbf{m} = (m_1, \ldots, m_d) \in \mathbb{N}^d$ *we say that* $\xi \in \mathbb{C}$ *is a system pole of order* τ *of* $(\mathbf{f}, \mathbf{m})$ *if* τ *is the largest positive integer such that for each* $s = 1, \ldots, \tau$ *there exists at least one polynomial combination of the form*

$$\sum_{k=1}^{d} p_k \mathfrak{f}_k, \qquad \deg p_k < m_k, \qquad k = 1, \ldots, d, \tag{4.1}$$

which is analytic on a neighborhood of $\overline{D}_{|\Phi_E(\xi)|}$ *except for a pole at* $z = \xi$ *of exact order* s.

Let τ be the order of ξ as a system pole of $\mathbf{f}$. For each $s = 1, \ldots, \tau$, let $\rho_{\xi,s}(\mathbf{f}, \mathbf{m})$ denote the largest of all the numbers $\rho_s(g)$ (the index of the largest canonical domain containing at most s poles of g), where g is a polynomial combination of type (4.1) that is holomorphic on a neighborhood of $\overline{D}_{|\Phi_E(\xi)|}$ except for a pole at $z = \xi$ of order s. Then, we define

$$R_{\xi,s}(\mathbf{f}, \mathbf{m}) := \min_{k=1,\ldots,s} \rho_{\xi,k}(\mathbf{f}, \mathbf{m}),$$

and

$$R_{\xi}(\mathbf{f}, \mathbf{m}) := R_{\xi,\tau}(\mathbf{f}, \mathbf{m}) = \min_{k=1,\ldots,\tau} \rho_{\xi,k}(\mathbf{f}, \mathbf{m}).$$

Fix $k = \{1, \ldots, d\}$. Let $D_k(\mathbf{f}, \mathbf{m})$ be the largest canonical domain in which all the poles of $\mathfrak{f}_k$ are system poles of $\mathbf{f}$ with respect to $\mathbf{m}$, their order as poles of $\mathfrak{f}_k$ does not exceed their order as system poles, and $\mathfrak{f}_k$ has no other singularity. By $R_k(\mathbf{f}, \mathbf{m})$, we denote the index of this canonical domain. Let $\xi_1, \ldots, \xi_N$ be the poles of $\mathfrak{f}_k$ in $D_k(\mathbf{f}, \mathbf{m})$. For each $j = 1, \ldots, N$, let $\hat{\tau}_j$ be the order of ξ_j as pole of f_k and τ_j its order as a system pole. By assumption, $\hat{\tau}_j \leq \tau_j$. Set

$$R_k^*(\mathbf{f}, \mathbf{m}) := \min \left\{ R_k(\mathbf{f}, \mathbf{m}), \min_{j=1,\ldots,N} R_{\xi_j, \hat{\tau}_j}(\mathbf{f}, \mathbf{m}) \right\}$$

and let $D_k^*(\mathbf{f}, \mathbf{m})$ be the canonical domain with this index.

By $Q_{\mathbf{m}}^{\mathbf{f}}$ we denote the monic polynomial whose zeros are the system poles of $\mathbf{f}$ with respect to $\mathbf{m}$ taking account of their order. The set of distinct zeros of $Q_{\mathbf{m}}^{\mathbf{f}}$ is denoted by $\mathcal{P}_{\mathbf{m}}^{\mathbf{f}}$.

The following theorem constitutes our main result in this chapter. It extends Theorem 1.2.1 to the case of MHP approximation.

Theorem 4.1.1. *Suppose (1.28) takes place. Let $\mathbf{f} \in \mathcal{H}(E)^d$ and fix a multi-index $\mathbf{m} \in \mathbb{N}^d$. Then, the next two assertions are equivalent:*

(a) $\mathbf{f}$ has exactly $|\mathbf{m}|$ system poles with respect to $\mathbf{m}$ counting multiplicities.

(b) For all sufficiently large n, the denominators $Q_{n,\mathbf{m}}$ of multipoint Hermite-Padé approximations of $\mathbf{f}$ are uniquely determined and there exists a polynomial $Q_{\mathbf{m}}$ of degree $|\mathbf{m}|$ such that

$$\limsup_{n \longrightarrow \infty} \|Q_{n,\mathbf{m}} - Q_{\mathbf{m}}\|^{1/n} = \theta < 1, \tag{4.2}$$

where $\|\cdot\|$ denotes the coefficient norm in the space of polynomials of degree $\leq |\mathbf{m}|$. Moreover, if either (a) or (b) takes place, then $Q_{\mathbf{m}} \equiv Q_{\mathbf{m}}^{\mathbf{f}}$,

$$\theta = \max \left\{ \frac{|\Phi_E(\xi)|}{R_\xi(\mathbf{f}, \mathbf{m})} : \xi \in \mathcal{P}_{\mathbf{m}}^{\mathbf{f}} \right\}, \tag{4.3}$$

and for any compact subset $\mathcal{K}$ of $D_k^*(\mathbf{f}, \mathbf{m}) \setminus \mathcal{P}_{\mathbf{m}}^{\mathbf{f}}$,

$$\limsup_{n \longrightarrow \infty} \|R_{n,\mathbf{m},k} - \mathfrak{f}_k\|_{\mathcal{K}}^{1/n} \leq \frac{\|\Phi_E\|_{\mathcal{K}}}{R_k^*(\mathbf{f}, \mathbf{m})}, \tag{4.4}$$

where $\|\cdot\|_{\mathcal{K}}$ denotes the sup-norm on $\mathcal{K}$ and if $\mathcal{K} \subset E$, then $\|\Phi_E\|_{\mathcal{K}}$ is replaced by 1.

§ 4.2. Direct statements.

For each $n \geq |\mathbf{m}|$, let $q_{n,\mathbf{m}}$ be the polynomial $Q_{n,\mathbf{m}}$ normalized so that

$$\sum_{k=0}^{|\mathbf{m}|} |\lambda_{n,k}| = 1, \qquad q_{n,\mathbf{m}}(z) = \sum_{k=0}^{|\mathbf{m}|} \lambda_{n,k} z^k. \tag{4.5}$$

This normalization implies that the polynomials $q_{n,\mathbf{m}}$ are uniformly bounded on each compact subset of $\mathbb{C}$.

***Lemma* 4.2.1.** *Assume that* (1.28) *takes place and let* ξ *be a system pole of order* τ *of* $\mathbf{f}$ *with respect to* $\mathbf{m}$. *Then*

$$\limsup_{n\to\infty} |q_{n,\mathbf{m}}^{(s)}(\xi)|^{1/n} \leq \frac{|\Phi_E(\xi)|}{R_{\xi,s+1}(\mathbf{f},\mathbf{m})}, \qquad s = 0,\dots,\tau-1. \tag{4.6}$$

Proof. Consider a polynomial combination g_1 of type (4.1) that is analytic on a neighborhood of $\overline{D}_{|\Phi_E(\xi)|}$ except for a simple pole $z=\xi$ and verifies that $\rho_1(g_1) = R_{\xi,1}(\mathbf{f},\mathbf{m}) (= \rho_{\xi,1}(\mathbf{f},\mathbf{m}))$. Then, we have

$$g_1 = \sum_{k=1}^{d} p_{k,1} \mathfrak{f}_k, \quad \deg p_{k,1} < m_k, \quad k = 1,\dots,d.$$

Define $h_1(z) = (z-\xi) g_1(z)$. The function

$$\frac{q_{n,\mathbf{m}}(z) h_1(z)}{a_{n+1}(z)} - \frac{z-\xi}{a_{n+1}(z)} \sum_{k=1}^{|\mathbf{m}|} p_{k,1}(z) P_{n,\mathbf{m},k}(z)$$

is analytic on $D_{\rho_1(g_1)}$. Take $1<\rho<\rho_1(g_1)$, and set $\Gamma_\rho = \{z : |\Phi_E(z)| = \rho\}$. Set $P_{n,1}(z) = \sum_{k=1}^{|\mathbf{m}|} p_{k,1}(z) P_{n,\mathbf{m},k}(z)$. Since $\deg(z-\xi)P_{n,1}(z) \leq n$, we have

$$\frac{1}{2\pi i} \int_{\Gamma_\rho} \frac{(t-\xi) P_{n,1}(t)}{(t-z) a_{n+1}(t)} dt = 0,$$

Using Hermite's interpolation formula (see [37]), we obtain

$$q_{n,\mathbf{m}}(z) h_1(z) - (z-\xi) \sum_{k=1}^{|\mathbf{m}|} p_{k,1} P_{n,\mathbf{m},k}(z) = \frac{1}{2\pi i} \int_{\Gamma_\rho} \frac{a_{n+1}(z)}{a_{n+1}(t)} \frac{q_{n,\mathbf{m}}(t) h_1(t)}{t-z} dt,$$

for all z with $|\Phi_E(z)| < \rho$. In particular, taking $z=\xi$ in the above formula, we arrive at

$$q_{n,\mathbf{m}}(\xi) h_1(\xi) = \frac{1}{2\pi i} \int_{\Gamma_\rho} \frac{a_{n+1}(\xi)}{a_{n+1}(t)} \frac{q_{n,\mathbf{m}}(t) h_1(t)}{t-\xi} dt. \tag{4.7}$$

Then, taking account of (1.28), it easily follows that

$$\limsup_{n\to\infty} |q_{n,\mathbf{m}}(\xi)h_1(\xi)|^{1/n} \le \frac{|\Phi_E(\xi)|}{\rho}.$$

Using that $h_1(\xi) \neq 0$ and making ρ tend to $\rho_1(g_1)$, we obtain

$$\limsup_{n\to\infty} |q_{n,\mathbf{m}}(\xi)|^{1/n} \le \frac{|\Phi_E(\xi)|}{R_{\xi,1}(\mathbf{f},\mathbf{m})} < 1.$$

Now, we employ induction. Suppose that

$$\limsup_{n\to\infty} |q^{(j)}_{n,\mathbf{m}}(\xi)|^{1/n} \le \frac{|\Phi_E(\xi)|}{R_{\xi,j+1}(\mathbf{f},\mathbf{m})}, \qquad j = 0,1,\ldots,s-2, \tag{4.8}$$

where $s \le \tau$. Let us prove that formula (4.8) holds for $j = s-1$. This will imply (4.6).

Consider a polynomial combination g_s of type (4.1) that is analytic on a neighborhood of $\overline{D}_{|\Phi_E(\xi)|}$ except for a pole of order s at $z = \xi$ and verifies that $\rho_s(g_s) = R_{\xi,s}(\mathbf{f},\mathbf{m})$. Then,

$$g_s = \sum_{k=1}^{|\mathbf{m}|} p_{k,s}\mathfrak{f}_k, \quad \deg p_{k,s} < m_k, \quad k = 1,\ldots,|\mathbf{m}|.$$

Set $h_s(z) = (z-\xi)^s g_s(z)$. The function

$$\frac{q_{n,\mathbf{m}}(z)h_s(z)}{a_{n+1}(z)(z-\xi)^{s-1}} - \frac{z-\xi}{a_{n+1}(z)}\sum_{k=1}^{|\mathbf{m}|} p_{k,s}(z)P_{n,\mathbf{m},k}(z)$$

is analytic on $D_{\rho_s(g_s)}\setminus\{\xi\}$. Set $P_{n,s} = \sum\limits_{k=1}^{|\mathbf{m}|} p_{k,s}P_{n,\mathbf{m},k}$. Fix an arbitrary compact set $\mathcal{K} \subset D_{\rho_s(g_s)}\setminus\{\xi\}$. Take $\delta > 0$ sufficiently small and $1 < \rho < \rho_s(g_s)$. Using Hermite's interpolation formula, for all $z \in \mathcal{K}$, we have

$$\frac{q_{n,\mathbf{m}}(z)h_s(z)}{(z-\xi)^{s-1}} - (z-\xi)P_{n,s}(z) = I_n(z) - J_n(z), \tag{4.9}$$

where

$$I_n(z) = \frac{1}{2\pi i}\int_{\Gamma_\rho} \frac{a_{n+1}(z)}{a_{n+1}(t)}\frac{q_{n,\mathbf{m}}(t)h_s(t)}{(t-\xi)^{s-1}(t-z)}dt$$

and

$$J_n(z) = \frac{1}{2\pi i}\int_{|t-\xi|=\delta} \frac{a_{n+1}(z)}{a_{n+1}(t)}\frac{q_{n,\mathbf{m}}(t)h_s(t)}{(t-\xi)^{s-1}(t-z)}dt.$$

The first integral I_n is estimated as in (4.7) to obtain

$$\limsup_{n\to\infty} \|I_n(z)\|_{\mathcal{K}}^{1/n} \leq \frac{\|\Phi_E\|_{\mathcal{K}}}{\rho_s(g_s)}. \tag{4.10}$$

For J_n, as $\deg q_{n,\mathbf{m}} \leq |\mathbf{m}|$ write

$$q_{n,\mathbf{m}}(t) = \sum_{j=0}^{|\mathbf{m}|} \frac{q_{n,\mathbf{m}}^{(j)}(\xi)}{j!}(t-\xi)^j.$$

Then

$$J_n(z) = \sum_{j=0}^{s-2} \frac{1}{2\pi i} \int_{|t-\xi|=\delta} \frac{a_{n+1}(z)}{a_{n+1}(t)} \frac{h_l(t)}{(t-\xi)^{s-1-j}} \frac{q_{n,\mathbf{m}}^{(j)}(\xi)}{j!(t-z)} dt. \tag{4.11}$$

Using the inductive hypothesis (4.8), from (4.11) it easily follows that

$$\limsup_{n\to\infty} \|J_n(z)\|_{\mathcal{K}}^{1/n} \leq \frac{\|\Phi_E\|_{\mathcal{K}}}{|\Phi_E(\xi)|} \frac{|\Phi_E(\xi)|}{R_{\xi,s-1}(\mathbf{f},\mathbf{m})} = \frac{\|\Phi_E\|_{\mathcal{K}}}{R_{\xi,s-1}(\mathbf{f},\mathbf{m})}. \tag{4.12}$$

Now, (4.9), (4.10), and (4.12) give

$$\limsup_{n\to\infty} \|q_{n,\mathbf{m}}(z)h_s(z) - (z-\xi)^s P_{n,s}(z)\|_{\mathcal{K}}^{1/n} \leq \frac{\|\Phi_E\|_{\mathcal{K}}}{R_{\xi,s}(\mathbf{f},\mathbf{m})}. \tag{4.13}$$

As the function inside the norm in (4.13) is analytic in $D_{\rho_l(g_l)}$, from the maximum principle it follows that (4.13) also holds for any compact set $\mathcal{K} \subset D_{\rho_l(g_l)}$. Using Cauchy's integral formula from (4.13) we also obtain that

$$\limsup_{n\to\infty} \|(q_{n,\mathbf{m}}(z)h_s(z) - (z-\xi)^s P_{n,s})^{(s-1)}(z)\|_{\mathcal{K}}^{1/n} \leq \frac{\|\Phi_E\|_{\mathcal{K}}}{R_{\xi,s}(\mathbf{f},\mathbf{m})}. \tag{4.14}$$

Taking $z = \xi$ in (4.14), we obtain

$$\limsup_{n\to\infty} |(q_{n,\mathbf{m}}h_s)^{(s-1)}(\xi)|^{1/n} \leq \frac{|\Phi_E(\xi)|}{R_{\xi,s}(\mathbf{f},\mathbf{m})}.$$

Using the Leibniz formula for higher derivatives of a product of two functions, the induction hypothesis (4.8), and that $h_s(\xi) \neq 0$, it follows that

$$\limsup_{n\to\infty} |q_{n,\mathbf{m}}^{(s-1)}(\xi)|^{1/n} \leq \frac{|\Phi_E(\xi)|}{R_{\xi,s}(\mathbf{f},\mathbf{m})},$$

This completes the induction. ∎

§ 4.2.1. Proof of $(a) \Rightarrow (b)$.

Let $\{\xi_1, \ldots, \xi_p\}$ be the distinct system poles of $\mathbf{f}$ with respect to $\mathbf{m}$, and let τ_j be the order of ξ_j as a system pole, $j = 1, \ldots, p$. By assumption, $\tau_1 + \cdots + \tau_p = |\mathbf{m}|$. We have proved that, for $j = 1, \ldots, p$ and $s = 0, 1, \ldots, \tau_j - 1$

$$\limsup_{n\to\infty} |q_{n,\mathbf{m}}^{(s)}(\xi_j)|^{1/n} \leq \frac{|\Phi_E(\xi_j)|}{R_{\xi_j,s+1}(\mathbf{f},\mathbf{m})} \leq \frac{|\Phi_E(\xi_j)|}{R_{\xi_j}(\mathbf{f},\mathbf{m})}, \tag{4.15}$$

where $R_{\xi_j}(\mathbf{f},\mathbf{m}) := R_{\xi_j,\tau}(\mathbf{f},\mathbf{m})$. Using Hermite interpolation, it is easy to construct a basis $\{\ell_{j,s}\}, 1 \leq j \leq p, 0 \leq s \leq \tau_j - 1$, in the space of polynomials of degree at most $|\mathbf{m}| - 1$ satisfying

$$\ell_{j,s}^{(k)}(\xi_i) = \delta_{i,j}\delta_{k,s}, \qquad 1 \leq i \leq p, \qquad 0 \leq k \leq \tau_i - 1.$$

Then,

$$q_{n,\mathbf{m}}(z) = \sum_{j=1}^{p} \sum_{s=0}^{\tau_j - 1} q_{n,\mathbf{m}}^{(s)}(\xi_j)\ell_{j,s}(z) + \lambda_{n,|\mathbf{m}|} Q_{\mathbf{m}}^{\mathbf{f}}. \tag{4.16}$$

Using (4.15) and (4.16), we have

$$\limsup_{n\to\infty} \|q_{n,\mathbf{m}} - \lambda_{n,|\mathbf{m}|} Q_{\mathbf{m}}^{\mathbf{f}}\|_{\mathcal{K}}^{1/n} \leq \theta \tag{4.17}$$

for any compact $\mathcal{K} \subset \mathbb{C}$, where

$$\theta = \max\left\{ \frac{|\Phi_E(\xi)|}{R_\xi(\mathbf{f},\mathbf{m})} : \ \xi \in \mathcal{P}_{\mathbf{m}}^{\mathbf{f}} \right\} < 1. \tag{4.18}$$

Now, necessarily

$$\liminf_{n\to\infty} |\lambda_{n,|\mathbf{m}|}| > 0. \tag{4.19}$$

Indeed, if there is a subsequence of indices $\Lambda \subset \mathbb{N}$ such that $\lim_{n\in\Lambda} |\lambda_{n,|\mathbf{m}|}| = 0$ then from (4.18), as the polynomials $q_{n,\mathbf{m}}$ converge, we would have that $\lim_{n\in\Lambda} q_{n,\mathbf{m}} = 0$ which contradicts (4.5). Since

$$q_{n,\mathbf{m}} = \lambda_{n,|\mathbf{m}|} Q_{n,\mathbf{m}}, \tag{4.20}$$

from (4.17) and (4.19) it follows that

$$\limsup_{n\to\infty} \|Q_{n,\mathbf{m}} - Q_{\mathbf{m}}^{\mathbf{f}}\|_{\mathcal{K}}^{1/n} \leq \theta. \tag{4.21}$$

In finite dimensional spaces all norms are equivalent; therefore, (4.21) is also true with the coefficient norm which means that (4.2) is satisfied with $=$ replaced by $\leq$.

In particular, for all sufficiently large n necessarily $\deg Q_{n,\mathbf{m}} = |\mathbf{m}|$. The difference of any two distinct monic polynomials satisfaying Definition 4.1.1 with the same degree produces a new solution of degree strictly less than $|\mathbf{m}|$, but we have proved that any solution must have degree $|\mathbf{m}|$ for all sufficiently large n. Hence, the polynomial $Q_{n,\mathbf{m}}$ is uniquely determined for all sufficiently large n.

Now, we prove the equality in (4.2). To the contrary, suppose that

$$\limsup_{n\to\infty} \|Q_{n,\mathbf{m}} - Q^{\mathbf{f}}_{\mathbf{m}}\|^{1/n} < \max\left\{\frac{|\Phi_E(\xi)|}{R_\xi(\mathbf{f},\mathbf{m})} : \xi \in \mathcal{P}^{\mathbf{f}}_{\mathbf{m}}\right\}. \tag{4.22}$$

Let ζ be a system pole of $\mathbf{f}$ such that

$$\frac{|\Phi_E(\zeta)|}{R_\zeta(\mathbf{f},\mathbf{m})} = \max\left\{\frac{|\Phi_E(\xi)|}{R_\xi(\mathbf{f},\mathbf{m})} : \xi \in \mathcal{P}^{\mathbf{f}}_{\mathbf{m}}\right\}. \tag{4.23}$$

Clearly, the inequality (4.22) implies that $R_\zeta(\mathbf{f},\mathbf{m}) < \infty$.

Choose a polynomial combination

$$g = \sum_{k=1}^{d} p_k \mathfrak{f}_k, \quad \deg p_k < m_k, \quad k = 1,\ldots,d, \tag{4.24}$$

that is holomorphic on a neighborhood of $\overline{D}_{|\Phi_E(\zeta)|}$ except for a pole of some order l at $z = \zeta$ with $\rho_l(g) = R_\zeta(\mathbf{f},\mathbf{m})$. On the boundary of $D_{\rho_l(g)}$, the function g must have a singularity which is not a system pole. In fact, if all the singularities were of this type, then we could find a different polynomial combination g_1 of type (4.24) for which $\rho_l(g_1) > \rho_l(g) = R_\zeta(\mathbf{f},\mathbf{m})$, which contradicts the definition of $R_\zeta(\mathbf{f},\mathbf{m})$. Therefore, $\rho_l(g) = \rho_0(Q^{\mathbf{f}}_{\mathbf{m}} g) = R_\zeta(\mathbf{f},\mathbf{m})$. Choose $1 < \rho < |\Phi_E(\zeta)|$. Using (1.29), we have

$$\frac{1}{R_\zeta(\mathbf{f},\mathbf{m})} = \limsup_{n\to\infty}\left|\int_{\Gamma_\rho} \frac{Q^{\mathbf{f}}_{\mathbf{m}}(t) g(t)}{a_{n+1}(t)} dt\right|^{1/n}. \tag{4.25}$$

Now,

$$\left(Q_{n,\mathbf{m}}(z) g(z) - \sum_{k=1}^{d} p_k(z) P_{n,\mathbf{m},k}(z)\right) / a_{n+1}(z)$$

is holomorphic in $D_{\rho_l(g)}$ and $\deg \sum_{k=1}^{d} p_k P_{n,\mathbf{m},k} < n$; therefore, from Cauchy's integral theorem we have that

$$0 = \int_{\Gamma_\rho} \frac{Q_{n,\mathbf{m}}(z) g(z) - \sum_{k=1}^{d} p_k(z) P_{n,\mathbf{m},k}(z)}{a_{n+1}(z)} dz = \int_{\Gamma_\rho} \frac{Q_{n,\mathbf{m}}(z) g(z)}{a_{n+1}(z)} dz \tag{4.26}$$

Combining (4.25) and (4.26), we get

$$\frac{1}{R_\zeta(\mathbf{f},\mathbf{m})} = \limsup_{n\to\infty} \left| \int_{\Gamma_\rho} \frac{g(t)}{a_{n+1}(t)} \left(Q^{\mathbf{f}}_{\mathbf{m}}(t) - Q_{n,\mathbf{m}}(t)\right) dt \right|^{1/n}. \tag{4.27}$$

This equality is impossible because from (1.28), (4.22), and (4.23) it is not hard to deduce that (4.27) is strictly less than $1/R_\zeta(\mathbf{f},\mathbf{m})$. This proves the equality in (4.2).

If ξ is any one of the system poles of $\mathbf{f}$ and τ its order, from (4.15) and (4.19), we have

$$\max_{j=0\ldots,l} \limsup_{n\to\infty} |Q^{(j)}_{n,\mathbf{m}}(\xi)|^{1/n} \leq \frac{|\Phi_E(\xi)|}{R_{\xi,l+1}(\mathbf{f},\mathbf{m})}, \quad l = 0, 1, \ldots, \tau - 1. \tag{4.28}$$

Now we are ready to prove (4.4). Let us fix $k \in \{1, \ldots, d\}$. Let $\mathcal{K}$ be a compact subset contained in $D^*_k(\mathbf{f},\mathbf{m}) \setminus \mathcal{P}^{\mathbf{f}}_{\mathbf{m}}$. Take $\delta > 0$ sufficiently small and $1 < \rho = R^*_k(\mathbf{f},\mathbf{m}) - \delta$, so that $\mathcal{K}$ lies in the region bounded by Γ_ρ and the circles $C_j = \{z : |t - \xi_j| = \delta\}$, $j = 1, \ldots, N_k$, where $\xi_1, \ldots, \xi_{N_k}$ are the poles of $\mathfrak{f}_k$ in $D^*_k(\mathbf{f},\mathbf{m})$. Let $\Gamma_{\rho,\delta}$ be the positively oriented curve determined by Γ_ρ and those circles C_j. On account of Definition 4.1.1, using Hermite's formula, we have

$$(Q_{n,\mathbf{m}}\mathfrak{f}_k - P_{n,\mathbf{m},k})(z) = \frac{1}{2\pi i} \int_{\Gamma_{\rho,\delta}} \frac{a_{n+1}(z)}{a_{n+1}(t)} \frac{(Q_{n,\mathbf{m}}\mathfrak{f}_k)(t)}{t - z} dt. \tag{4.29}$$

From (1.28) it readily follows that

$$\limsup_{n\to\infty} \left| \frac{1}{2\pi i} \int_{\Gamma_\rho} \frac{a_{n+1}(z)}{a_{n+1}(t)} \frac{(Q_{n,\mathbf{m}}\mathfrak{f}_k)(t)}{t - z} dt \right|^{1/n} \leq \frac{\|\Phi_E\|_{\mathcal{K}}}{R^*_k(\mathbf{f},\mathbf{m})} \tag{4.30}$$

Let $\hat{\tau}_j$ be the order of ξ_j as pole of $\mathfrak{f}_k$. Using the expansion

$$Q_{n,\mathbf{m}}(t) = \sum_{l=0}^{|\mathbf{m}|} \frac{Q^{(l)}_{n,\mathbf{m}}(\xi_j)}{l!} (t - \xi_j)^l,$$

for the circle C_j we have

$$\frac{1}{2\pi i} \int_{C_j} \frac{a_{n+1}(z)}{a_{n+1}(t)} \frac{(Q_{n,\mathbf{m}}\mathfrak{f}_k)(t)}{t - z} dt = \sum_{l=0}^{\hat{\tau}_j - 1} \frac{1}{2\pi i} \int_{C_j} \frac{a_{n+1}(z)}{a_{n+1}(t)} \frac{(t - \xi_j)^{\hat{\tau}_j} \mathfrak{f}_k(t)}{(t - \xi)^{\hat{\tau}_j - l}} \frac{Q^{(j)}_{n,\mathbf{m}}(\xi)}{l!(t - z)} dt \tag{4.31}$$

because the function under the integral sign is analytic inside C_j for $\hat{\tau}_j \leq l \leq |\mathbf{m}|$. Now, (1.28) and (4.28) allow to deduce from (4.31) that

$$\limsup_{n\to\infty}\left|\frac{1}{2\pi i}\int_{C_j}\frac{a_{n+1}(z)}{a_{n+1}(t)}\frac{(Q_{n,\mathbf{m}}\mathfrak{f}_k)(t)}{t-z}dt\right|^{1/n} \leq \frac{\|\Phi_E\|_{\mathcal{K}}}{|\Phi_E(\xi_j)|}\frac{|\Phi_E(\xi_j)|}{R_{\xi_j,\hat{\tau}_j}(\mathbf{f},\mathbf{m})}. \quad (4.32)$$

Finally, (4.29), (4.30), and (4.32) give (4.4). □

A slight variation of the arguments employed above allow to deduce the following corollary of independent interest.

***Corollary* 4.2.1.** *Suppose that* (1.28) *takes place and* $\mathbf{f}$ *has exactly* $|\mathbf{m}|$ *system poles with respect to* $\mathbf{m}$. *Then, for every system pole* ξ *of* $\mathbf{f}$

$$\max_{j=0\ldots,l}\limsup_{n\to\infty}|Q^{(j)}_{n,\mathbf{m}}(\xi)|^{1/n} = \frac{|\Phi_E(\xi)|}{R_{\xi,l+1}(\mathbf{f},\mathbf{m})}, \qquad l = 0,1,\ldots,\tau-1. \quad (4.33)$$

where τ *is the order of* ξ.

Proof. If (4.33) fails, due to (4.28), there is a system pole ξ of $\mathbf{f}$ of order τ such that for some $l, 0 \leq l < \tau$

$$\max_{j=0\ldots,l}\limsup_{n\to\infty}|Q^{(j)}_{n,\mathbf{m}}(\xi)|^{1/n} < \frac{|\Phi_E(\xi)|}{R_{\xi,l+1}(\mathbf{f},\mathbf{m})}. \quad (4.34)$$

Now, we argue by contradiction as in the proof of the equality in (4.2).

Choose a polynomial combination as in (4.24) that is analytic on a neighborhood of $\overline{D}_{|\Phi_E(\xi)|}$ except for a pole of order $s(\leq l+1)$ at $z = \xi$ with $\rho_s(g) = R_{\xi,s}(\mathbf{f},\mathbf{m}) \geq R_{\xi,l+1}(\mathbf{f},\mathbf{m})$. On the boundary of $D_{\rho_s(g)}$, the function g must have a singularity which is not a system pole. Set $Q^{\mathbf{f}}_{\mathbf{m}} = Q_{\mathbf{m}}$. Take $\delta > 0$ sufficiently small and $1 < \rho < \rho_s(g)$. Let $\Gamma_{\rho,\delta}$ be the positively oriented curve determined by Γ_ρ and $\{t : |t-\xi| = \delta\}$. According to (1.29)

$$\frac{1}{\rho_s(g)} = \limsup_{n\to\infty}\left|\int_{\Gamma_{\rho,\delta}}\frac{Q_{\mathbf{m}}(t)g(t)}{a_{n+1}(t)}dt\right|^{1/n}. \quad (4.35)$$

The function

$$\frac{H_n(z)}{a_{n+1}(z)} = \frac{Q_{n,\mathbf{m}}(z)g(z) - \sum_{k=1}^{|\mathbf{m}|}p_k(z)P_{n,\mathbf{m},k}(z)}{a_{n+1}(z)}$$

is analytic in $D_{\rho_s(g)} \setminus \{\xi\}$ and

$$\int_{\Gamma_{\rho,\delta}} \frac{H_n(t)}{a_{n+1}(t)} dt = 0$$

Set $P_n = \sum\limits_{k=1}^{d} p_k P_{n,\mathbf{m},k}$ and $h(t) = (t-\xi)^s g(t)$. Obviously,

$$Q_{\mathbf{m}} g = (Q_{\mathbf{m}} - Q_{n,\mathbf{m}})g + P_n + H_n,$$

and since $\deg P_n \leq n-1$, we obtain

$$\begin{aligned}
&\int_{\Gamma_{\rho,\delta}} \frac{Q_{\mathbf{m}}(t)g(t)}{a_{n+1}(t)} dt = \int_{\Gamma_{\rho,\delta}} \frac{[Q_{\mathbf{m}} - Q_{n,\mathbf{m}}](t)h(t)}{(t-\xi)^s a_{n+1}(t)} dt \\
&= \int_{\Gamma_{\rho}} \frac{[Q_{\mathbf{m}} - Q_{n,\mathbf{m}}](t)h(t)}{(t-\xi)^s a_{n+1}(t)} dt - \sum_{j=0}^{|\mathbf{m}|} \int_{|t-\xi|=\delta} \frac{[Q_{\mathbf{m}}^{(j)} - Q_{n,\mathbf{m}}^{(j)}](\xi)h(t)}{j!(t-\xi)^{s-j} a_{n+1}(t)} dt \\
&= \int_{\Gamma_{\rho}} \frac{[Q_{\mathbf{m}} - Q_{n,\mathbf{m}}](t)h(t)}{(t-\xi)^s a_{n+1}(t)} dt + \sum_{j=0}^{s-1} \int_{|t-\xi|=\delta} \frac{Q_{n,\mathbf{m}}^{(j)}(\xi)h(t)}{j!(t-\xi)^{s-j} a_{n+1}(t)} dt.
\end{aligned}$$

Estimating these integrals, using (1.28), (4.3) and the assumption (4.34) it is easy to deduce that

$$\limsup_{n\to\infty} \left| \int_{\Gamma_{\rho,\delta}} \frac{Q_{\mathbf{m}}(t)g(t)}{a_{n+1}(t)} dt \right|^{1/n} < \frac{1}{\rho_s(g)}$$

which contradicts (4.35). Therefore, (4.34) cannot occur and there is equality in (4.33). ■

§ 4.3. Inverse statements.

§ 4.3.1. Some auxiliary results.

It is easy to see that Buslaev's theorem remains valid also for Laurent series $\mathfrak{f}(z) = \sum_{-\infty}^{\infty} f_n z^n$ such that $\limsup_{n\to\infty} |f_{-n}|^{1/n} \leq R_0(\mathfrak{f})$. Moreover, Buslaev's theorem can be supplemented by the following assertion (see [3]).

Supplement to Buslaev's Theorem. Suppose that the power series (1.6) is not a polynomial, $R_0(\mathfrak{f}) = \infty$, and

$$\alpha_{n,0} f_n + \alpha_{n,-1} f_{n+1} + \cdots = 0 \quad (n = 1, 2, \dots) \tag{4.36}$$

where the $\alpha_n(z) = \sum_{p=0}^{\infty} \alpha_{n,-p} z^{-p}$ $(n = 1, 2, \dots)$ are holomorphic and converge to $\alpha(z)$ in the exterior of some disk as $n \to \infty$. Then $\alpha(\infty) = 0$, and the coefficients $\{f_n\}$ of the series (1.6) satisfy

$$\epsilon_{n,0} f_n + \cdots + \epsilon_{n,-N+1} f_{n+N-1} + f_{n+N} = 0, \quad \lim_n \epsilon_{n,p} = \epsilon_p, \tag{4.37}$$

where $p = 1, \dots, k$ and N is the multiplicity of the zero of $\alpha(z)$ at $z = \infty$.

This result will be useful in the next section to prove Lemma 4.3.1.

§ 4.3.2. Incomplete multipoint Padé approximants.

Let us introduce the notion of incomplete multipoint Padé approximants, proving results of inverse type. Similar concepts proved to be effective in the study of Hermite-Padé approximation, see [9] and [10].

***Definition* 4.3.1.** *Let $\mathfrak{f} \in \mathcal{H}(E)$. Fix $m \geq m^* \geq 1$, $n \geq m$. We say that the rational function $R_{n,m}$ is an incomplete multipoint Padé approximation of type (n, m, m^*) corresponding to $\mathfrak{f}$ if $R_{n,m}$ is the quotient of any two polynomials $P_{n,m}$, $Q_{n,m}$ that verify*

c.1) $\deg P_{n,m} \leq n - m^*$, $\deg Q_{n,m} \leq m$, $Q_{n,m} \not\equiv 0$,

c.2) $\frac{Q_{n,m}\mathfrak{f} - P_{n,m}}{a_{n+1}} \in \mathcal{H}(E)$

where $a_n(z) = \prod_{k=1}^{n} (z - \alpha_{n,k})$.

Since $Q_{n,m} \not\equiv 0$, we normalize it to be monic. We call $Q_{n,m}$ the denominator of the corresponding (n, m, m^*) incomplete multipoint Padé approximant of $\mathfrak{f}$. Notice that for each $k = 1, \dots, d$, the polynomial $Q_{n,\mathbf{m}}$ given in Definition 4.1.1, is a denominator of an $(n, |\mathbf{m}|, m_k)$ incomplete multipoint Padé approximant for $\mathfrak{f}_k$.

In this section, we will study the relation between the convergence of $Q_{n,m}$ and some analytic properties of $\mathfrak{f}$.

***Lemma* 4.3.1.** *Let $\mathfrak{f} \in \mathcal{H}(E)$ and fix $m \geq m^* \geq 1$. Suppose that $\mathfrak{f}$ is not a rational function with al most $m^* - 1$ poles and there exists a polynomial Q_m of degree m such that*

$$\limsup_{n\to\infty} \|Q_{n,m} - Q_m\|^{1/n} \leq \theta < 1. \tag{4.38}$$

Then, either $\mathfrak{f}$ has exactly m^* poles in $D_{\rho_{m^*}(\mathfrak{f})}$ or $\rho_0(Q_m\mathfrak{f}) > \rho_{m^*}(\mathfrak{f})$.

Proof. For each $n \geq m$, let $q_{n,m}$ be the polynomial $Q_{n,m}$ normalized so that

$$\sum_{k=0}^{|\mathbf{m}|} |\lambda_{n,k}| = 1, \qquad q_{n,m}(z) = \sum_{k=0}^{|\mathbf{m}|} \lambda_{n,k} z^k. \tag{4.39}$$

Let $\{\xi_1, \ldots, \xi_\omega\}$ be the distinct poles of $\mathfrak{f}$ in $D_{\rho_{m^*}(\mathfrak{f})}$ and $\tau_1, \ldots, \tau_\omega$ be their orders, respectively. Consequently,

$$\sum_{j=1}^{\omega} \tau_j \leq m^*.$$

Modifying conveniently the proof of (4.6), we can show that for $j = 1, \ldots, \omega$ and $\nu = 0, 1, \ldots, \tau_j - 1$,

$$\limsup_{n\to\infty} |q_{n,m}^{(\nu)}(\xi_j)|^{1/n} \leq \frac{\Phi(\xi_j)}{\rho_{m^*}(f)} < 1. \tag{4.40}$$

Since the sequence of polynomials $Q_{n,m}$ converges to Q_m, (4.40) entails that ξ_j is a zero of Q_m of multiplicity at least τ_j. Being this the case, for each pole of $\mathfrak{f}$ in $D_{\rho_{m^*}(\mathfrak{f})}$, we have

$$\rho_0(Q_m\mathfrak{f}) \geq \rho_{m^*}(\mathfrak{f}).$$

Should $\rho_0(Q_m\mathfrak{f}) > \rho_{m^*}(\mathfrak{f})$ we are done.

Suppose that $\rho_0(Q_m\mathfrak{f}) = \rho_{m^*}(\mathfrak{f})$. To conclude the proof, let us show that in this situation $\mathfrak{f}$ has exactly m^* poles in $D_{\rho_{m^*}(\mathfrak{f})}$. To the contrary, suppose that $\mathfrak{f}$ has in $D_{\rho_{m^*}(\mathfrak{f})}$ at most $m^* - 1$ poles. Then, there exists a polynomial $\deg Q_{m^*} < m^*$ such that

$$\rho_0(Q_{m^*}\mathfrak{f}) = \rho_{m^*}(\mathfrak{f}) = \rho_0(Q_m Q_{m^*}\mathfrak{f}).$$

It follows from Definition 4.3.1 that

$$\frac{Q_{m^*}(Q_{n,m}\mathfrak{f} - P_{n,m})}{a_{n+1}} \in \mathcal{H}(E).$$

Then

$$\int_{\Gamma_\rho} \frac{Q_{m^*}(z)(Q_{n,m}\mathfrak{f} - P_{n,m})(z)}{a_{n+1}(z)} dz = 0,$$

where Γ_ρ is a contour encircling E and lying in the domain of holomorphy of $\mathfrak{f}(z)$. Since each one of the $n+1$ zeros of the polynomial $a_{n+1}(z)$ lies on E and $\deg(Q_{m^*}P_{n,m})(z) \leq n-1$, it follows that

$$\int_{\Gamma_\rho} \frac{Q_{m^*}(z)P_{n,m}(z)}{a_{n+1}(z)} dz = 0.$$

Therefore,
$$\int_{\Gamma_\rho} \frac{Q_{m^*}(z)Q_{n,m}(z)\mathfrak{f}(z)}{a_{n+1}(z)}dz = 0. \tag{4.41}$$
Take $1 < \rho < \rho_{m^*}(\mathfrak{f})$. Then, by (1.29)
$$\frac{1}{\rho_{m^*}(\mathfrak{f})} = \frac{1}{\rho_0(Q_m Q_{m^*}\mathfrak{f})} = \limsup_{n\to\infty} \left| \int_{\Gamma_\rho} \frac{(Q_m Q_{m^*}\mathfrak{f})(t)}{a_{n+1}(t)}dt \right|^{1/n}$$
$$= \limsup_{n\to\infty} \left| \int_{\Gamma_\rho} \frac{(Q_{m^*}\mathfrak{f})(t)}{a_{n+1}(t)} \left(Q_{n,m} - Q_m\right)(t)dt \right|^{1/n}.$$
Using (1.28) and (4.38) to estimate the last integral, it readily follows that
$$\frac{1}{\rho_{m^*}(\mathfrak{f})} \le \frac{\theta}{\rho_{m^*}(\mathfrak{f})}, \qquad \theta < 1,$$
which implies that $\rho_{m^*}(\mathfrak{f}) = \infty$. Now, let us show that this is not possible.

Take $F(w) = Q_{m^*}(\varphi(w))\mathfrak{f}(\varphi(w))$, where $\varphi = \psi_E^{-1}$. Let γ be a contour encircling $\{w : |w| = 1\}$ lying in the domain of holomorphy of $F(w)$. Using (4.41), we obtain
$$0 = \int_\gamma \frac{F(w)Q_{n,m}(\varphi(w))}{a_{n+1}(\varphi(w))}\varphi'(w)dw =$$
$$\int_\gamma F(w)\frac{Q_{n,m}(\varphi(w))}{w^m}\frac{w^{n+1}}{a_{n+1}(\varphi(w))}\varphi'(w)\frac{dw}{w^{n+1-m}}$$
Setting
$$\alpha_n(w) = \frac{Q_{n,m}(\varphi(w))}{w^m}\frac{(cw)^{n+1}}{a_{n+1}(\varphi(w))}\varphi'(w),$$
the previous inequality means that
$$\left[F(w)\alpha_n(w)\right]_{n-m} = 0. \tag{4.42}$$
The functions $\alpha_n(w)$ $(n = 1, 2, \dots)$ are holomorphic in the exterior of the unit disk (including $w = \infty$) and, due to (1.28) and (4.3.1), converge as $n \to \infty$ to
$$\alpha(w) = \varphi'(w)\frac{Q_m(\varphi(w))}{w^m G(\varphi(w))} = \sum_{p=0}^{\infty} \alpha_{-p}w^{-p}, \qquad \alpha_0 = \alpha(\infty) \ne 0.$$

Let $\sum_{-\infty}^{\infty} F_n w^n$ be the Laurent expansion of the function F outside the unit circle, i.e:

$$F(w) = \sum_{-\infty}^{\infty} F_n w^n = F_1(w) + F_2(w),$$

where $F_1(w) = \sum_0^{\infty} F_n w^n$. Then, $R_0(F_1) = \infty$ and (4.42) holds (for all sufficiently large n) replacing F with F_1. According to the Supplement to Buslaev's Theorem and the fact that $\alpha(\infty) \neq 0$, we get that F_1 must be a polynomial. Consequently, F is either analytic or has a pole at ∞. In turn this implies that $Q_{m^*}\mathfrak{f}$ is either analytic or has a pole at ∞. However, $Q_{m^*}\mathfrak{f}$ is an entire function because it is holomorphic in $\mathbb{C}$ since $R_0(Q_{m^*}\mathfrak{f}) = \infty$. Therefore, $Q_{m^*}\mathfrak{f}$ is a polynomial, or what is the same $\mathfrak{f}$ is a rational function with at most $m^* - 1$ poles against our hypothesis on $\mathfrak{f}$. This contradiction implies that the assumption that $\mathfrak{f}$ had in $D_{\rho_{m^*}(\mathfrak{f})}$ at most $m^* - 1$ poles is false. So the number of poles on $\mathfrak{f}$ in $D_{\rho_{m^*}(\mathfrak{f})}$ must equal m^* as claimed. ■

§ 4.3.3. Polynomial independence.

Let us introduce the concept of polynomial independence of a vector of functions.

***Definition* 4.3.2.** *A vector* $\mathbf{f} = (\mathfrak{f}_1, \ldots, \mathfrak{f}_d) \in \mathcal{H}(E)^d$ *is said to be polynomially independent with respect to* $\mathbf{m} = (m_1, \ldots, m_d) \in \mathbb{N}^d$ *if there do not exist polynomials* $p_1, \ldots, p_d$*, at least one of which is non-null, such that*

(i) $\deg p_k < m_k$*;* $\quad k = 1, \ldots, d,$

(ii) $\sum_{i=k}^{d} p_k \mathfrak{f}_k$ *is a polynomial.*

In particular, polynomial independence implies that for each $k = 1, \ldots, d$, $\mathfrak{f}_k$ is not a rational function with at most $m_k - 1$ poles.

***Lemma* 4.3.2.** *Let* $\mathbf{f} \in \mathcal{H}(E)^d$ *and fix a multi-index* $\mathbf{m} \in \mathbb{N}^d$*. Suppose that for all* $n \geq n_0$*, the polynomial* $Q_{n,\mathbf{m}}$ *is unique and* $\deg Q_{n,\mathbf{m}} = |\mathbf{m}|$*. Then the system* $\mathbf{f}$ *is polynomially independent with respect to* $\mathbf{m}$*.*

Proof. Except for a small detail, the proof coincides with that of [10, Lemma 3.2]. Given $\mathbf{f} := (\mathfrak{f}_1, \ldots, \mathfrak{f}_d)$ and $\mathbf{m} := (m_1, \ldots, m_d) \in \mathbb{N}^d$, we consider the associated system

$$\overline{\mathbf{f}} := (\mathfrak{f}_1, \ldots, z^{m_1 - 1}\mathfrak{f}_1, \mathfrak{f}_2, \ldots, z^{m_d - 1}\mathfrak{f}_d) = (\bar{\mathfrak{f}}_1, \ldots, \bar{\mathfrak{f}}_{|\mathbf{m}|}).$$

We also define an associated multi-index $\overline{\mathbf{m}} := (1, \ldots, 1)$ with $|\mathbf{m}| = |\overline{\mathbf{m}}|$. The systems $\mathbf{f}$ and $\overline{\mathbf{f}}$ share most properties. In particular, poles and system poles of $(\mathbf{f}, \mathbf{m})$ and $(\overline{\mathbf{f}}, \overline{\mathbf{m}})$ coincide. Then, we can assume without loss generality that $\mathbf{m} = (1, \ldots, 1)$ and $d = |\mathbf{m}|$.

Suppose that there exist constants c_k, $k = 1, \ldots, d$, not all zero, such that $\sum_{k=1}^{d} c_k \mathfrak{f}_k$ is a polynomial. Without loss of generality, we can assume that $c_1 \neq 0$. Then,

$$\mathfrak{f}_1 = p - \sum_{k=2}^{d} c_k f_k,$$

where p is a polynomial of degree N.

On other hand, for each $n \geq d - 1$, there exist polynomials Q_n, $P_{n,k}$, $k = 2, \ldots, d$, such that

- $\deg P_{n,k} \leq n - 1$, $\deg Q_n \leq d - 1$, $Q_n \not\equiv 0$,
- $\frac{Q_n \mathfrak{f}_k - P_{n,k}}{a_{n+1}} \in \mathcal{H}(E)$.

Therefore,

$$\frac{Q_n\left(p - \sum_{k=2}^{d} c_k \mathfrak{f}_k\right) - \left(Q_n p - \sum_{k=2}^{d} c_k P_{n,k}\right)}{a_{n+1}} \in \mathcal{H}(E)$$

and, for $n \geq d + N$, the polynomial $P_{n,1} = Q_n p - \sum_{k=2}^{d} c_k P_{n,k}$ verifies $\deg P_{n,1} \leq n - 1$. Thus, for all n sufficiently large, the polynomials $P_{n,k}$, $k = 1, \ldots, d$ satisfy Definition 4.1.1 with respect to $\mathbf{f}$ and $\mathbf{m}$. Naturally, Q_n gives rise to a polynomial $Q_{n,\mathbf{m}}$ with $\deg Q_{n,\mathbf{m}} < d = |\mathbf{m}|$ against our assumption on $Q_{n,\mathbf{m}}$. ■

The following corollary is a straightforward consequence of Lemma 4.3.1.

***Corollary* 4.3.1.** *Let $\mathbf{f} \in \mathcal{H}(E)^d$ and fix a multi-index $\mathbf{m} \in \mathbb{N}^d$. Assume that $\mathbf{f}$ is polynomially independent with respect to $\mathbf{m}$ and there exists a polynomial $Q_{\mathbf{m}}$ of degree $|\mathbf{m}|$ such that.*

$$\limsup_{n\to\infty} \|Q_{n,\mathbf{m}} - Q_{\mathbf{m}}\|^{1/n} \leq \theta < 1. \tag{4.43}$$

Then for each $k = 1, \ldots, d$, either $\mathfrak{f}_k$ has exactly m_k poles in $D_{\rho_{m_k}(\mathfrak{f}_k)}$ or $\rho_0(Q_{\mathbf{m}}\mathfrak{f}_k) > \rho_{m_k}(\mathfrak{f}_k)$.

Before proving the inverse statements of Theorem 4.1.1, we wish to describe some properties of system poles. For the proof see [10, Lemma 3.5].

***Lemma* 4.3.3.** *Let $\mathbf{f} \in \mathcal{H}(E)^d$ and $\mathbf{m} \in \mathbb{N}^d$, $\mathbf{f}$ can have at most $|\mathbf{m}|$ system poles with respect to $\mathbf{m}$ (counting their order). Moreover, if the system $\mathbf{f}$ has exactly $|\mathbf{m}|$ system poles with respect to $\mathbf{m}$ and ξ is a system pole of order τ, then for all $s > \tau$ there can be no polynomial combination of the form* (4.1) *holomorphic on a neighborhood of $\overline{D}_{|\Phi(\xi)|}$ except for a pole at $z = \xi$ of exact order s.*

§ 4.3.4. Proof $(b) \Rightarrow (a)$.

Given $\mathbf{f} := (\mathfrak{f}_1, \dots, \mathfrak{f}_d)$ and $\mathbf{m} := (m_1, \dots, m_d) \in \mathbb{N}^d$, we consider the associated system

$$\bar{\mathbf{f}} := (\mathfrak{f}_1, \dots, z^{m_1-1}\mathfrak{f}_1, \mathfrak{f}_2, \dots, z^{m_d-1}\mathfrak{f}_d) = (\bar{\mathfrak{f}}_1, \dots, \bar{\mathfrak{f}}_{|\mathbf{m}|}).$$

We also define an associated multi-index $\overline{\mathbf{m}} := (1, 1, \dots, 1)$ with $|\mathbf{m}| = |\overline{\mathbf{m}}|$. From Definition 4.3.2, it readily follows that $\mathbf{f}$ is polynomially independent with respect to $\mathbf{m}$ if and only if there do not exist constant c_k, $k = 1, \dots, |\mathbf{m}|$, not all zero, such that

$$\sum_{k=1}^{|\mathbf{m}|} c_k \bar{\mathfrak{f}}_k$$

is a polynomial. That is, $\mathbf{f}$ is polynomially independent with respect to $\mathbf{m}$ if and only if $\bar{\mathbf{f}}$ is polynomially independent with respect to $\overline{\mathbf{m}}$. Moreover, due to Lemma 4.3.2, on account of the hypothesis we know that $\mathbf{f}$ is polynomially independent with respect to $\mathbf{m}$ which implies that $\bar{\mathbf{f}}$ is polynomially independent with respect to $\overline{\mathbf{m}}$. So, it is enough to prove that $\mathbf{f}$ has exactly $|\mathbf{m}|$ system poles with respect to $\mathbf{m}$ and without loss of generality we can assume that $\mathbf{m} = (1, 1, \dots, 1)$.

The auxiliary results that we have obtained thus far allow us to adapt the proof employed in [10] to obtain a similar result for the case of row sequences of Hermite-Padé approximations. For completeness we include the whole proof.

The plan is as follows. First, we collect a set of $|\mathbf{m}|$ candidates to be system poles of $\mathbf{f}$ (counting their orders) and prove that they are zeros of $Q_{\mathbf{m}}$. In the second part we prove that all these points previously selected are actually system poles of $\mathbf{f}$.

Notice that for each $k = 1, \dots, |\mathbf{m}|$, by Corollary 4.3.1, either $D_{\rho_1(\mathfrak{f}_k)}$ contains exactly one pole of $\mathfrak{f}_k$ and it is a zero of $Q_{\mathbf{m}}$, or $\rho_0(Q_{\mathbf{m}}\mathfrak{f}_k) > \rho_1(\mathfrak{f}_k)$. Hence, $D_{\rho_0(\mathbf{f})} \neq \mathbb{C}$ and $Q_{\mathbf{m}}$ contains as zeros all the poles of $\mathfrak{f}_k$ on the boundary of $D_{\rho_0(\mathfrak{f}_k)}$ counting their order for $k = 1, \dots, |\mathbf{m}|$. Moreover, the function

$\mathfrak{f}_k$ cannot have on the boundary of $D_{\rho_0(\mathfrak{f}_k)}$ singularities other than poles. Hence, the poles of $\mathbf{f}$ on the boundary of $D_{\rho_0(\mathbf{f})}$ are all zeros of $Q_{\mathbf{m}}$ counting multiplicities and the boundary contains no other singularity except poles. Let us call them candidate system poles of $\mathbf{f}$ and denote them by $a_1, \ldots, a_{n_1}$ repeated according to their order. They constitute the first layer of candidate system poles of $\mathbf{f}$.

Since $\deg Q_{\mathbf{m}} = |\mathbf{m}|$, $n_1 \leq |\mathbf{m}|$. If $n_1 = |\mathbf{m}|$, we are done. Let us assume that $n_1 < |\mathbf{m}|$ and let us find coefficients $c_1, \ldots, c_{|\mathbf{m}|}$ such that

$$\sum_{k=1}^{|\mathbf{m}|} c_k \mathfrak{f}_k$$

is holomorphic in a neighborhood of $\overline{D}_{\rho_0(\mathbf{f})}$. Finding those $c_1, \ldots, c_{|\mathbf{m}|}$ reduces to solving a homogeneous system of n_1 linear equations with $|\mathbf{m}|$ unknowns. In fact, if $z = a$ is a candidate system pole of $\mathbf{f}$ with multiplicity τ, we obtain τ equations choosing the coefficients c_k so that

$$\int_{|\omega-a|=\delta} (\omega - a)^k \left(\sum_{k=1}^{|\mathbf{m}|} c_k \mathfrak{f}_k(\omega) \right) = 0, \quad k = 0, \ldots, \tau - 1. \tag{4.44}$$

We have the same type of system of equations for each distinct candidate system pole on the boundary of $D_{\rho_0(\mathbf{f})}$. Combining these equations, we obtain a homogeneous system of n_1 linear equations with $|\mathbf{m}|$ unknowns. Moreover, this homogeneous system of linear equations has at least $|\mathbf{m}| - n_1$ linearly independent solutions, which we denote by $\mathbf{c}_j^1$, $j = 1, \ldots, |\mathbf{m}| - n_1^*$, where $n_1^* \leq n_1$ denotes the rank of the system of equations.

Let

$$\mathbf{c}_j^1 := (c_{j,1}^1, \ldots, c_{j,|\mathbf{m}|}^1), \qquad j = 1, \ldots, |\mathbf{m}| - n_1^*.$$

Define the $(|\mathbf{m}| - n_1^*) \times |\mathbf{m}|$ dimensional matrix

$$C^1 := \begin{pmatrix} c_1^1 \\ \vdots \\ c_{|\mathbf{m}|-n_1^*}^1 \end{pmatrix}.$$

Define the vector $\mathbf{g}_1$ of $|\mathbf{m}| - n_1^*$ functions given by

$$\mathbf{g}_1^t := C^1 \mathbf{f}^t = (g_{1,1}, \ldots, g_{1,|\mathbf{m}|-n_1^*})^t,$$

where $(\cdot)^t$ means taking transpose. Since all the rows of C^1 are non-null and $\mathbf{f}$ is polynomially independent with respect to $\mathbf{m}$, none of the functions

$$g_{1,j} = \sum_{k=1}^{|\mathbf{m}|} c^1_{j,k}\mathfrak{f}_k, \qquad j = 1, \ldots, |\mathbf{m}| - n_1^*,$$

are polynomials.

Consider the canonical domain

$$D_{\rho_0(g_1)} = \bigcap_{j=1}^{|\mathbf{m}|-n_1^*} D_{\rho_0(g_{1,j})}.$$

Obviously, by construction, $D_{\rho_0(\mathbf{f})}$ is strictly included in $D_{\rho_0(g_1)}$. Therefore, for each $j = 1, \ldots, |\mathbf{m}|-n_1^*$, $Q_{n,\mathbf{m}}$ is a denominator of an $(n, |\mathbf{m}|, 1)$ multipoint incomplete Padé approximant of $g_{1,j}$. Since all $g_{1,j}$ are not polynomials, by Lemma 4.3.1 with $m^* = 1$, for each $j = 1, \ldots, |\mathbf{m}| - n_1^*$, either $D_{\rho_1(g_{1,j})}$ contains exactly one pole of $g_{1,j}$ and it is a zero of $Q_{\mathbf{m}}$, or $\rho_0(Q_{\mathbf{m}} g_{1,j}) > \rho_1(g_{1,j})$. In particular, $D_{\rho_0(\mathbf{g_1})} \neq \mathbb{C}$ and all the singularities of $\mathbf{g_1}$ on the boundary of $D_{\rho_0(\mathbf{g_1})}$ are poles which are zeros of $Q_{\mathbf{m}}$ counting their order. They form the next layer of candidate system pole of $\mathbf{f}$.

Denote by $a_{n_1+1}, \ldots, a_{n_1+n_2}$ these new candidate system poles. Again, if $n_1 + n_2 = |\mathbf{m}|$, we are done. Otherwise, $n_2 < |\mathbf{m}| - n_1 \leq |\mathbf{m}| - n_1^*$, and we keep repeating the same process by eliminating the n_2 poles $a_{n_1+1}, \ldots, a_{n_1+n_2}$. In order to do that, we have $|\mathbf{m}| - n_1^*$ functions which are holomorphic in $D_{\rho(\mathbf{g_1})}$ and meromorphic on a neighborhood of $\overline{D}_{\rho(\mathbf{g_1})}$. The corresponding homogeneous system of linear equations, similar to (4.44), has al least $|\mathbf{m}| - n_1^* - n_2^*$ linearly independent solutions $\mathbf{c}^2_j$, where $n_2^* \leq n_2$ is the rank of the new system. Let

$$\mathbf{c}^2_j := (c^2_{j,1}, \ldots, c^2_{j,|\mathbf{m}|}), \quad j = 1, \ldots, |\mathbf{m}| - n_1^* - n_2^*.$$

Define the $(|\mathbf{m}| - n_1^* - n_2^*) \times |\mathbf{m} - n_1^*|$ dimensional matrix

$$C^2 := \begin{pmatrix} c^2_1 \\ \vdots \\ c^2_{|\mathbf{m}|-n_1^*-n_2^*} \end{pmatrix}.$$

Define the vector $\mathbf{g}_2$ of $|\mathbf{m}| - n_1^* - n_2^*$ functions given by

$$\mathbf{g}_2^t := C^2\mathbf{g}_1^t = C^2C^1\mathbf{f}^t = (g_{2,1}, \ldots, g_{2,|\mathbf{m}|-n_1^*-n_2^*})^t.$$

It is a basic fact from linear algebra that if C_1 has full rank and C_2 has full rank, then C_2C_1 has full rank. This means that the rows of C_2C_1 are linearly independent, particulary, they are non-null. Therefore, none of the component functions of $\mathbf{g}_2$ are polynomials because of the polynomial independence of $\mathbf{f}$ with respect to $\mathbf{m}$. Thus, we can apply again Lemma 4.3.1. Using finite induction, we find a total on $|\mathbf{m}|$ candidate system poles.

In fact, on each layer of system poles, $n_k \geq 1$. Therefore, in a finite number of steps, say $N-1$, their sum equals to $|\mathbf{m}|$. Consequently, the number of candidate system poles of $\mathbf{f}$ in some disk, counting multiplicities, is exactly equal to $|\mathbf{m}|$, and they are precisely the zeros of $Q_{|\mathbf{m}|}$ as we wanted to prove.

Summarizing, in the $N-1$ steps we have taken, we have produced N layers of candidate system poles. Each layer contains n_k candidates, $k = 1, \ldots, N$. At the same time, on each step k, $k = 1, \ldots, N-1$, we have solved a system of n_k linear equations, of rank n_k^*, with $|\mathbf{m}| - n_1^* - \cdots - n_k^*$, $n_k^* \leq n_k$, linearly independent solutions. We find ourselves on the N-th layer with n_N candidates.

Let us try to eliminate the poles on the last layer. Write the corresponding homogeneous system of liner equations as in (4.44), and we get n_N equations where

$$n_N = |\mathbf{m}| - n_1 - \cdots - n_{N-1} \leq |\mathbf{m}| - n_1^* - \cdots - n_{N-1}^* =: \overline{n}_N$$

with $\overline{n}_N$ unknowns. For each candidate system pole a of multiplicity τ on the N-th layer, we impose the equations

$$\int_{|\omega-a|=\delta} (\omega-a)^i \left(\sum_{k=1}^{\overline{n}_N} c_k g_{N-1,k}(\omega) \right) = 0, \quad i = 0, \ldots, \tau-1, \tag{4.45}$$

where δ is sufficiently small and the $g_{N-1,k}$, $k = 1, \ldots, \overline{n}_N$, are the functions associated with the linearly independent solutions produced on step $N-1$.

Let n_N^* be the rank of this last homogeneous system of linear equations. Assume that $n_N^* < n_k$ for some $k \in \{1, \ldots, N\}$. Then, the rank of the last system of equations is strictly less than the number of unknowns, namely $n_N^* < \overline{n}_N$. Therefore, repeating the same process, there exists a vector of functions

$$\mathbf{g}_N := (g_{N,1}, \ldots, g_{N,|\mathbf{m}|-n_1^*-\cdots-n_N^*})$$

such that none of the $g_{N,k}$ is a polynomial because of the polynomial independence of $|\mathbf{f}|$ with respect to $\mathbf{m}$. Applying Lemma 4.3.1, each $g_{N,k}$ has on the boundary of its disk of analyticity a pole which is a zero of $Q_{\mathbf{m}}$. However, this is impossible because all the zeros of $Q_{\mathbf{m}}$ are strictly contained in that disk. Consequently, $n_k^* = n_k$, $k = 1, \ldots, N$.

We conclude that all the N homogeneous systems of linear equations that we have solved have full rank. This implies that if in any one of those N systems of equations we equate one of its equations to 1 instead of zero (see (4.44) or (4.45)), the corresponding nonhomogeneous system of linear equations has a solution. By the definition of a system pole, this implies that each candidate system pole is indeed a system pole of order at least equal to its multiplicity as zero of $Q_{\mathbf{m}}$. Moreover, by Lemma 4.3.3, $\mathbf{f}$ can have al most $|\mathbf{m}|$ system pole with respect to $\mathbf{m}$; therefore, all candidate system poles are system poles, and their order coincides with the multiplicity of that point as a zero of $Q_{\mathbf{m}}$. This also means that $Q_{\mathbf{m}} = Q_{\mathbf{m}}^{\mathbf{f}}$. Thus, the proof of the inverse type result is complete. $\square$

CHAPTER 5

Anexos

§ 5.1. Numerical examples.

Example 1: Fix $\mathbf{m} = (1, 1)$ and take $\mathbf{f} = (\mathfrak{f}_1, \mathfrak{f}_2)$ where

$$\mathfrak{f}_1(z) = \frac{1}{z-1} + e^z \quad \text{and} \quad \mathfrak{f}_2(z) = \frac{1}{z-1} + \log(2-z).$$

We get that $q_{\mathbf{m}}(z) = (z-1)(z/2-1)$ but one sequence of zeros converges very fast to 1 whereas the other one converges slowly to 2, we can see it in Table 5.1. The $q_k^*(z)$ denote the polynomials in this case of degree less than or equal to 1 which appear in (1.22). There is a polynomial $q_k^*(z)$ for each function of the system $\mathfrak{f}_k$, in this example $k = 1, 2$.

Valor de n	$q_{n,\mathbf{m}}(z) = 0$	$q_1^*(z) = 0$	$q_2^*(z) = 0$
$n = 5$	$z_1 \approx 3.5362$ $z_2 \approx 1.0027$	$z \approx 0.4580$	$z \approx 2.2513$
$n = 100$	$z_1 \approx 2.0206$ $z_2 = 1.0$	$z \approx 2.0203$	$z = 1.0$
$n = 150$	$z_1 \approx 2.0136$ $z_2 = 1.0$	$z \approx 2.0135$	$z = 1.0$
$n = 200$	$z_1 \approx 2.0101$ $z_2 = 1.0$	$z \approx 2.0100$	$z = 1.0$
$n = 500$	$z_1 \approx 2.004024$ $z_2 = 1.0$	$z \approx 2.004015$	$z = 1.0$

Table 5.1: Example 1

Example 2: Fix $\mathbf{m} = (1, 1)$ and take $\mathbf{f} = (\mathfrak{f}_1, \mathfrak{f}_2)$ where

$$\mathfrak{f}_1(z) = \frac{1}{z-1} + e^z \quad \text{and} \quad \mathfrak{f}_2(z) = \frac{1}{z-1} + \log(3-z).$$

We get that $q_{\mathbf{m}}(z) = (z-1)(z/3-1)$ but one sequence of zeros converges very fast to 1 whereas the other one converges slowly to 3, we can see it in Table 5.2. The $q_k^*(z)$ denote the polynomials in this case of degree less than or equal to 1 which appear in (1.22). There is a polynomial $q_k^*(z)$ for each function of the system $\mathfrak{f}_k$, in this example $k = 1, 2$.

Valor de n	$q_{n,\mathbf{m}}(z) = 0$	$q_1^*(z) = 0$	$q_2^*(z) = 0$
$n = 5$	$z_1 \approx 3.7717$ $z_2 \approx 0.9997$	$z \approx 1.1275$	$z \approx -3.7432$
$n = 100$	$z_1 \approx 3.0307$ $z_2 = 1.0$	$z \approx 3.0304$	$z = 1.0$
$n = 150$	$z_1 \approx 3.0203$ $z_2 = 1.0$	$z \approx 3.0201$	$z = 1.0$
$n = 200$	$z_1 \approx 3.01518$ $z_2 = 1.0$	$z \approx 3.01511$	$z = 1.0$
$n = 500$	$z_1 \approx 3.00603$ $z_2 = 1.0$	$z \approx 3.00601$	$z = 1.0$

Table 5.2: Example 2

Example 3: Fix $\mathbf{m} = (1, 1, 1)$ and take $\mathbf{f} = (\mathfrak{f}_1, \mathfrak{f}_2)$ where

$$\mathfrak{f}_1(z) = \log(1-z) \quad \text{and} \quad \mathfrak{f}_2(z) = \frac{1}{z-1} + e^z.$$

We get that $q_{\mathbf{m}}(z) = (z-1)^2$ but one sequence of zeros converges very fast to 1 whereas the other one does it slowly, we can see it in Table 5.3. The $q_k^*(z)$ denote the polynomials in this case of degree less than or equal to 1 which appear in (1.22). There is a polynomial $q_k^*(z)$ for each function of the system $\mathfrak{f}_k$, in this example $k = 1, 2$.

Valor de n	$q_{n,\mathbf{m}}(z) = 0$	$q_1^*(z) = 0$	$q_2^*(z) = 0$
$n = 5$	$z_1 \approx 1.8401$ $z_2 \approx 1.0734$	$z \approx 0.7906$	$z \approx 1.3776$
$n = 10$	$z_1 \approx 1.2500$ $z_2 \approx 1.0000$	$z \approx 0.9994$	$z \approx 1.2177$
$n = 100$	$z_1 \approx 1.0204$ $z_2 = 1.0$	$z = 1.0$	$z \approx 1.0201$
$n = 150$	$z_1 \approx 1.0135$ $z_2 = 1.0$	$z = 1.0$	$z \approx 1.0134$
$n = 200$	$z_1 \approx 1.0101$ $z_2 = 1.0$	$z = 1.0$	$z \approx 1.0100$
$n = 500$	$z_1 \approx 1.00401$ $z_2 = 1.0$	$z = 1.0$	$z \approx 1.00400$

Table 5.3: Example 3

Example 4: Fix $\mathbf{m} = (1, 1, 1)$ and take $\mathbf{f} = (\mathfrak{f}_1, \mathfrak{f}_2, \mathfrak{f}_3)$ where

$$\mathfrak{f}_1(z) = \frac{1}{z-1} + e^z \quad \mathfrak{f}_2(z) = \log(1-z) \quad \text{and} \quad \mathfrak{f}_3(z) = \frac{1}{z+1} + e^z.$$

We get that $q_{\mathbf{m}}(z) = (z-1)^2(z+1)$, that is, two sequences of zeros converge to 1, one of them very fast whereas the other one does it slowly and one sequence converges very fast to -1, we can see it in Table 5.4. The $q_k^*(z)$ denote the polynomials in this case of degree less than or equal to 2 which appear in (1.22). There is a polynomial $q_k^*(z)$ for each function of the system $\mathfrak{f}_k$, in this example $k = 1, 2, 3$.

Valor de n	$q_{n,\mathbf{m}}(z)=0$	$q_1^*(z)=0$	$q_2^*(z)=0$	$q_3^*(z)=0$
$n=10$	$z_1\approx 1.2702$ $z_2\approx 1.0003$ $z_3\approx -0.9999$	$z_1\approx 1.2314$ $z_2\approx -0.9999$	$z_1\approx 0.9978$ $z_2\approx -0.9999$	$z_1\approx 1.2316$ $z_2\approx 1.0000$
$n=100$	$z_1\approx 1.0205$ $z_2=1.0$ $z_3=-1.0$	$z_1\approx 1.0203$ $z_2\approx -1.0$	$z_1=1.0$ $z_2=-1.0$	$z_1\approx 1.0203$ $z_2=1.0$
$n=150$	$z_1\approx 1.0135$ $z_2=1.0$ $z_3=-1.0$	$z_1\approx 1.0134$ $z_2=-1.0$	$z_1=1.0$ $z_2=-1.0$	$z_1\approx 1.0134$ $z_2=1.0$
$n=200$	$z_1\approx 1.01012$ $z_2=1.0$ $z_3=-1.0$	$z_1\approx 1.0100$ $z_2=-1.0$	$z_1=1.0$ $z_2=-1.0$	$z_1\approx 1.0100$ $z_2=1.0$
$n=500$	$z_1\approx 1.0040$ $z_2=1.0$ $z_3=-1.0$	$z_1\approx 1.00401$ $z_2=-1.0$	$z_1=1.0$ $z_2=-1.0$	$z_1\approx 1.0040$ $z_2=1.0$

Table 5.4: Example 4

Example 5: Fix $\mathbf{m}=(1,1,1)$ and take $\mathbf{f}=(\mathfrak{f}_1,\mathfrak{f}_2,\mathfrak{f}_3)$ where

$$\mathfrak{f}_1(z)=\frac{1}{z-1}+e^z \quad \mathfrak{f}_2(z)=\log(1-z) \quad \text{and} \quad \mathfrak{f}_3(z)=\frac{1}{z-2}+e^z.$$

We get that $q_{\mathbf{m}}(z)=(z-1)^2(z-2)$, that is, two sequences of zeros converge to 1 one of them very fast whereas the other one does it slowly and one sequence converges very fast to 2, we can see it in Table 5.6. The $q_k^*(z)$ denote the polynomials in this case of degree less than or equal to 2 which appear in (1.22). There is a polynomial $q_k^*(z)$ for each function of the system $\mathfrak{f}_k$, in this example $k=1,2,3$.

Valor de n	$q_{n,\mathbf{m}}(z)=0$	$q_1^*(z)=0$	$q_2^*(z)=0$	$q_3^*(z)=0$
$n=10$	$z_1\approx 1.7213$ $z_2\approx 1.1544$ $z_3\approx 0.9989$	$z_1\approx 1.9977$ $z_2\approx 1.1763$	$z_1\approx 0.9887$ $z_2\approx 0.1676$	$z_1\approx 1.1763$ $z_2\approx 1.0000$
$n=100$	$z_1\approx 2.0$ $z_2\approx 1.0201$ $z_3\approx 1.0$	$z_1\approx 2.0$ $z_2\approx 1.0199$	$z_1=2.0$ $z_2=1.0$	$z_1\approx 1.0199$ $z_2=1.0$
$n=150$	$z_1\approx 2.0$ $z_2\approx 1.0134$ $z_3\approx 1.0$	$z_1=2.0$ $z_2\approx 1.0133$	$z_1=2.0$ $z_2=1.0$	$z_1\approx 1.0133$ $z_2=1.0$
$n=200$	$z_1\approx 2.0$ $z_2\approx 1.0100$ $z_3\approx 1.0$	$z_1=2.0$ $z_2\approx 1.0099$	$z_1=2.0$ $z_2=1.0$	$z_1\approx 1.0099$ $z_2=1.0$
$n=500$	$z_1\approx 2.0$ $z_2\approx 1.0040$ $z_3\approx 1.0$	$z_1=2.0$ $z_2\approx 1.0039$	$z_1=2.0$ $z_2=1.0$	$z_1\approx 1.0039$ $z_2=1.0$

Table 5.5: Example 5

Example 6: Fix $\mathbf{m}=(1,1,1)$ and take $\mathbf{f}=(\mathfrak{f}_1,\mathfrak{f}_2,\mathfrak{f}_3)$ where

$$\mathfrak{f}_1(z)=\frac{1}{z-1}+e^z \quad \mathfrak{f}_2(z)=\log(1-z) \quad \text{and} \quad \mathfrak{f}_3(z)=\frac{1}{(z-1)^3}+e^z.$$

Put $\widehat{\mathbf{f}}=(\mathfrak{f}_1,\mathfrak{f}_2,\mathfrak{f}_4)$ where $\mathfrak{f}_4=\mathfrak{f}_1-\mathfrak{f}_3$. The $(n,\mathbf{m})$ Hermite-Padé approximants of the systems $\mathbf{f}$ and $\widehat{\mathbf{f}}$ have the same common denominator $q_{n,\mathbf{m}}$. We obtain the numerical results to the system $\widehat{\mathbf{f}}$ in Table **??**. We get that $q_{\mathbf{m}}(z)=(z-1)^3$ but one sequence of zeros converges very fast to 1 whereas the other two do it slowly. The $q_k^*(z)$ denote the polynomials in this case of degree less than or equal to 2 which appear in (1.22). There is a polynomial $q_k^*(z)$ for each function of the system $\mathfrak{f}_k$, in this example $k=1,2,3$.

Valor de n	$q_{n,\mathbf{m}}(z) = 0$	$q_1^*(z) = 0$	$q_2^*(z) = 0$	$q_3^*(z) = 0$
$n = 10$	$z_1 \approx 1.3913$ $z_2 \approx 1.0042$ $z_3 \approx 0.9241$	$z_1 \approx 1.3291$ $z_2 \approx 0.9341$	$z_1 \approx 0.9670 + 0.018i$ $z_2 \approx 0.967 - 0.018i$	$z_1 \approx 1.5981$ $z_2 \approx 1.0199$
$n = 100$	$z_1 \approx 1.0281$ $z_2 = 1.0$ $z_3 \approx 0.9926$	$z_1 \approx 1.0278$ $z_2 \approx 0.9927$	$z_1 = 1.0$ $z_2 = 0.995$	$z_1 \approx 1.0412$ $z_2 = 1.0$
$n = 150$	$z_1 \approx 1.0185$ $z_2 \approx 1.0$ $z_3 \approx 0.9951$	$z_1 \approx 1.0184$ $z_2 \approx 0.9951$	$z_1 = 1.0$ $z_2 \approx 0.9966$	$z_1 \approx 1.0272$ $z_2 = 1.0$
$n = 200$	$z_1 \approx 1.0138$ $z_2 \approx 1.0$ $z_3 \approx 0.9963$	$z_1 \approx 1.0137$ $z_2 \approx 0.9963$	$z_1 = 1.0$ $z_2 \approx 0.9975$	$z_1 \approx 1.0203$ $z_2 = 1.0$
$n = 500$	$z_1 \approx 1.0054$ $z_2 \approx 1.0$ $z_3 \approx 0.9985$	$z_1 \approx 1.0054$ $z_2 \approx 0.9985$	$z_1 = 1.0$ $z_2 = 0.999$	$z_1 \approx 1.0080$ $z_2 = 1.0$

Table 5.6: Example 6

Example 7: Fix $\mathbf{m} = (1, 1, 1)$ and take $\mathbf{f} = (\mathfrak{f}_1, \mathfrak{f}_2, \mathfrak{f}_3)$ where

$$\mathfrak{f}_1(z) = \frac{1}{z-1} + e^z \quad \mathfrak{f}_2(z) = \frac{1}{(z-1)^2} + e^z \quad \text{and} \quad \mathfrak{f}_3(z) = \log(1-z).$$

Put $\widehat{\mathbf{f}} = (\mathfrak{f}_1, \mathfrak{f}_4, \mathfrak{f}_3)$ where $\mathfrak{f}_4 = \mathfrak{f}_1 - \mathfrak{f}_2$. The $(n, \mathbf{m})$ Hermite-Padé approximants of the systems $\mathbf{f}$ and $\widehat{\mathbf{f}}$ have the same common denominator $q_{n,\mathbf{m}}$. We obtain the numerical results to the system $\widehat{\mathbf{f}}$ in Table **??**. We get that $q_{\mathbf{m}}(z) = (z-1)^3$ but one sequences of zeros converges very fast to 1 whereas the other one does it slowly. The $q_k^*(z)$ denote the polynomials in this case of degree less than or equal to 2 which appear in (1.22). There is a polynomial $q_k^*(z)$ for each function of the system $\mathfrak{f}_k$, in this example $k = 1, 2, 3$.

Valor de n	$q_{n,m}(z) = 0$	$q_1^*(z) = 0$	$q_2^*(z) = 0$	$q_3^*(z) = 0$
$n = 10$	$z_1 \approx 1.4348$ $z_2 \approx 1.0189$ $z_3 \approx 0.9850$	$z_1 \approx 1.3632$ $z_2 \approx 1.0000$	$z_1 \approx 7.5572$ $z_2 \approx 1.3630$	$z_1 \approx 0.9852 + 0.0268i$ $z_2 \approx 0.9852 - 0.0268i$
$n = 100$	$z_1 \approx 1.0309$ $z_2 \approx 1.0$ $z_3 \approx 1.0$	$z_1 \approx 1.0306$ $z_2 \approx 1.0$	$z_1 \approx 97.969$ $z_2 \approx 1.0306$	$z_1 \approx 1.0$ $z_2 \approx 1.0$
$n = 150$	$z_1 \approx 1.0204$ $z_2 \approx 1.0$ $z_3 \approx 1.0$	$z_1 \approx 1.0202$ $z_2 \approx 1.0$	$z_1 \approx 147.97$ $z_2 \approx 1.0202$	$z_1 \approx 1.0$ $z_2 \approx 1.0$
$n = 200$	$z_1 \approx 1.0152$ $z_2 \approx 1.0$ $z_3 \approx 1.0$	$z_1 \approx 1.0151$ $z_2 \approx 1.0$	$z_1 \approx 197.98$ $z_2 \approx 1.0151$	$z_1 \approx 1.0$ $z_2 \approx 1.0$
$n = 500$	$z_1 \approx 1.0060$ $z_2 \approx 1.0$ $z_3 \approx 1.0$	$z_1 \approx 1.0060$ $z_2 \approx 1.0$	$z_1 \approx 497.99$ $z_2 \approx 1.0060$	$z_1 \approx 1.0$ $z_2 \approx 1.0$

Table 5.7: Example 7

Bibliography

[1] S. Agmon. Sur les series de Dirichlet. Ann. Sci. École Norm. Sup. **66** (1949), 263-310.

[2] G. Baker and P.R. Graves-Morris. Padé Approximants, Parts I and II. Encycl. Math. Vols. 13 and 14. Cambridge University Press, Cambridge, 1981.

[3] V. I. Buslaev. Relations for the coefficients, and singular points of a function. Math. USSR Sb. **59** (1988), 349-377.

[4] V.I. Buslaev. An analogue of Fabry's theorem for generalized Padé approximants. Sb. Math. 200 (2009), 39-106.

[5] L. Bieberbach. Analytische Fortsetzung. Springer-Verlag, 1955.

[6] N. Bosuwan and G. Lopez Lagomasino. Determining system poles using row sequences of orthogonal Hermite-Padé approximants. J. Approx. Theory 231 (2018), 15-40.

[7] N. Bosuwan and G. Lopez Lagomasino. Direct and inverse results on row sequences of simultaneous Padé-Faber approximants. Mediterranean Journal of Mathematics (2019) on-line.

[8] N. Bosuwan, G. López Lagomasino, and Y. Zaldivar Gerpe. Direct and inverse results for multipoint Hermite-Padé approximants. arxiv 1810.07061 (accepted for publication in Journal of Analysis and Mathematical Physics).

[9] J. Cacoq, B. de la Calle Ysern, and G. López Lagomasino. Incomplete Padé approximation and convergence of row sequences of Hermite-Padé approximants. J. Approx. Theory **170** (2013), 59-77.

[10] J. Cacoq, B. de la Calle Ysern, and G. López Lagomasino. Direct and inverse results on row sequences of Hermite-Padé approximants. Constr. Approx, **38** (2013), 133-160.

[11] J. Coates. On the algebraic approximation of functions. I, II, III. Indag. Math. 28 (1966), 421-461.

[12] M.A. Evgrafov. A new proof of Perron's theorem. Izv. Akad. Nauk SSSR Ser. Math. **17** (1953), 77-82.

[13] E. Fabry. Sur les points singuliers d'une fonction donnée par son dévelopement de Taylor. Ann. Ec. Norm. Sup. Paris **13** (1896), 367-399.

[14] A. O. Gel'fond. Calculus of Finite Differences. International monographs on advanced mathematics and physics. Hindustan Pub. Corp., Delhi, 1971.

[15] A. A. Gonchar. On convergence of Padé approximants for some classes of meromorphic functions. Math. USSR Sb. **26** (1975), 555-575.

[16] A. A. Gonchar. On the convergence of generalized Padé approximants of meromorphic functions. Math. USSR Sb. **27** (1975), 503-514.

[17] A. A. Gonchar. Poles of rows of the Padé table and meromorphic continuation of functions. Sb. Math. **43** (1982), 527-546.

[18] A. A. Gonchar. Rational approximation of analytic functions. Proc. Steklov Inst. Math. **272** (2011), S44-S57.

[19] Ch. Hermite. Sur la fonction exponentielle, C. R. Acad. Sci. Paris 77 (1873), 18-24, 74-79, 226-233, 285-293; reprinted in his Oeuvres, Tome III, Gauthier-Villars, Paris, 1912, 150-181.

[20] H. Jager. A simultaneous generalization of the Padé table. I-VI. Indag. Math. 26 (1964), 193-249.

[21] G. López Lagomasino and Y. Zaldivar Gerpe. Inverse results on row sequences of Hermite-Padé approximation. Proc. Steklov Inst. Math. 298 (2017), 152-169.

[22] G. López Lagomasino and Y. Zaldivar Gerpe. Higher order recurrences and row sequences of Hermite-Padé approximation. Journal of Difference Equations and Applications, 24:11 (2018), 1830-1845.

[23] H. Poincaré. Sur les equations linéaires aux différentielles ordinaires et aux différences finies. Amer. J. Math. **7** (1885), 203-258.

[24] P. R. Graves-Morris and E.B. Saff. A de Montessus theorem for vector-valued rational interpolants. Lecture Notes in Math. Vol. 1105, 227-242, Springer, Berlin, 1984.

[25] J. Hadamard. Essai sur l'étude des fonctions données par leur développement de Taylor. J.Math. Pures Appl. **8** (1892), 101-186.

[26] G. López Lagomasino. On row sequences of Padé and Hermite-Padé approximation. Contemporary Mathematics, Vol. 661, 141-152, Amer. Math. Soc., Providence, R.I., 2016.

[27] K. Mahler. Perfect systems. Compos. Math. 19 (1968), 95-166.

[28] S. Mandelbrojt. Dirichlet Series. Principles and Methods. Reidel Pub. Co., Dordrecht, 1972.

[29] R. de Montessus de Ballore. Sur les fractions continues algébriques. Bull. Soc. Math. France **30** (1902), 28-36.

[30] O. Perron. Uber eine Satz des Herrn Ponccaré. J. Reine Angew. Math. **136** (1909), 17-37.

[31] O. Perron. Uber die Poincaresche lineare Differenzengleichung. J. Reine Angew. Math. **137** (1910), 6-64.

[32] A. Sidi. A de Montessus type convergence study of a least-squares vector-valued rational interpolation procedure II. Computational Methods and Function Theory **10** (2010), 223-247.

[33] S. P. Suetin. On poles of the mth row of a Padé table. Math. USSR Sb **48** (1984), 493-497.

[34] S. P. Suetin. On an inverse problem for the mth row of the Padé table. Math. USSR Sb. **52** (1985), 231-244.

[35] S. P. Suetin. Padé approximation and efficient analytic continuation of a power series. Russian Math. Surveys **53** (2002), 43-141.

[36] M. Van Barel and A. Bultheel. A new approach to the rational interpolation problem: the vector case. J. Comput. Appl. Math. 33 (1990) 331-346.

[37] J.L. Walsh. Interpolation and Approximation by Rational Functions in the Complex Domain. Coll. Pub XXI Amer. Soc. Providence, R.I., (1956).